CONTENTS FOR FEBRUARY, 1949

PAPERS

PAGE

Property Surveys Must Fit Their Titles.
By William C. Wattles 161

Continuous Arches and Bents Analyzed by Column Analogy.
By T. M. Yu 167

Analysis of Pile Foundations with Batter Piles.
By A. Hrennikoff 185

Photogrammetric Space Resection.
By Carl Chow 209

Wind Stresses in Semi-Rigid Connections of Steel Framework.
By Basil Sourochnikoff 235

DISCUSSIONS

The Significance of Pore Pressure in Hydraulic Structures.
By Paul E. Raes 247

Experiences with Predetermining Pile Lengths.
By H. Cambefort 251

Shearing Stress Distribution in Box Girders with Multiple Webs.
By A. S. Neiman 257

Panama Canal—The Sea-Level Project: A Symposium.
By Ernest Shankland, and John S. McNown 259

Valuation and Depreciation of Public Utility Property.
By Anston Marston, and James P. McKean 263

Flexural Constants of Haunched Beams by Area Computation.
By Robert V. Hauer 272

A Simplified Method of Analyzing Suspension Bridges.
By Pedro B. J. Gravina, and G. S. Ramaswamy 275

The Logical System of Freight Rates.
By Otto Meyer 282

Industrial Wastes: A Symposium.
By S. Leary Jones 283

CONTENTS FOR FEBRUARY, 1949 (*Continued*)

GUIDEPOST FOR TECHNICAL READERS
FEBRUARY, 1949, *PROCEEDINGS*

Items of value or significance to readers in various fields are here listed. For convenience the distribution is given in terms of the Society's Technical Divisions

Technical Division	*Pages*
Air Transport	161, 185, 209, 235
City Planning	161, 209, 263, 282
Construction	185, 235, 251, 259
Engineering Economics	263, 282
Highway	161, 185, 275, 282
Hydraulics	247, 263
Irrigation	247
Power	247, 263
Sanitary Engineering	247, 283
Soil Mechanics and Foundations	185, 247, 251
Structural	167, 185, 235, 247, 257, 259, 272, 275
Surveying	161, 209
Waterways	247, 251, 259

(A constant effort is made to supply technical material to Society members, over the entire range of possible interest. In so far as your specialty may be covered inadequately in the foregoing list, this fact is a gage of the need for your help toward improvement.—Ed.)

HINTS TO AUTHORS

Those who are planning papers for submission to PROCEEDINGS *will expedite Committee action measurably by first studying the standard instructions as to style, content, and format. For reprints, address the Manager, Technical Publications, 33 W. 39th St., New York 18, N. Y.*

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

PAPERS

PROPERTY SURVEYS MUST FIT THEIR TITLES

By William C. Wattles[1]

Synopsis

The basic purpose of this paper is to emphasize the fact that land surveying is a specialized profession. The relation between a land title and a property survey is discussed, thus leading into a statement of steps preceding a field survey, a brief description of office and field relocation methods, the legal status and value of physical monuments, and the tracing of land titles from record.

Property boundary location is an art and profession as distinct and different from civil engineering as dentistry is distinct from medicine. The mere fact that the mechanics of surveying, in the use of transit, tape, and computing tables, is common to both is no criterion of identity. The fact that a person is versed in the tactics of surveying as taught in school or college does not warrant the assumption that he is a land surveyor any more than that he is a civil engineer.

Many people hold the idea that the surveying art is a mechanical adjunct to engineering procedure, rather than an informative or determinative function. This is particularly true in the case of property surveys. Often an owner will endeavor to ascertain certain desired relations between title and possession by all the available methods other than survey, even though such survey is the cheapest, quickest, and, frequently, the only method of solution.

That surveying, notably in measurements of land, is a specialized occupation is evidenced historically by early records in Egypt, Rome, and elsewhere. The surveys of the land holdings along the Nile and the special courses of education for surveys in Rome are cases in point. No land location project of modern times is originated or completed without an accurate boundary survey.

It is self-evident that the value of a property survey is correlated with, and dependent on, the ownership or title of the land under consideration.

Land ownership presupposes a valid claim of a proprietor in the physical possession, actual or potential, of the area in question. The claim is further supported by a written record, describing the boundaries of the land in such a

Note.—Written comments are invited for immediate publication; to insure publication the last discussion should be submitted by **July 1, 1949.**

[1] Glendale, Calif.; formerly Chf. Engr., Title Ins. and Trust Co., Los Angeles, Calif.

manner that they may be recognized and, if necessary, be relocated in their original position. Specifically then, the physical location and the record description must be conformable and consistent to provide a correct title in the boundary. The accomplishment of this function is the province of the land surveyors.

The correct definition of boundaries of a parcel of land includes:

1. Examination of title ownership of record;
2. Claimant ownership of the ground area by occupation or other evidence;
3. Monumentation, both physical and as recorded;
4. Deeds, maps, descriptions, and other documentary evidence;
5. Adjoiner agreement or conflict;
6. Surveys of the instant and associated lands;
7. Abstracts of title or title insurance policies; and
8. Any other matters connected with the property in question.

No accurate survey of such land can be made without considering and weighing all the foregoing items. In addition, the surveyor must perform his work, both field and office, to mechanical and mathematical standards commensurate with the best practice. As a chain is no stronger than its weakest link, so is a land location survey no better than the error caused by omission or neglect of any of the factors entering into the determination.

The written record of a land holding is fixed and permanent; the physical location and monumentation are subject to change or destruction from many causes. In most cases, therefore, it becomes the duty of the surveyor to reestablish the record as nearly as possible, and to indicate its variance with the existing occupation. The legal questions of ownership or perference between positions of record, or physical claim, must be decided by the courts; it is the surveyor's province to point out the differences.

The location of a parcel having arbitrarily determined boundaries is comparatively simple, requiring only accurate dimensional treatment and a correctly established and proved tie point referenced to record. Whether the field work precedes or succeeds the description for record, the creating of one from the other is readily accomplished; and with reasonable care, the ground work will agree with the description. Slight variations discovered in relocation are nominally assignable to ordinary natural causes such as earth movement of various kinds or to the common personal errors of observation and measurement of the surveyor. The acceptance and use of monuments in such a case will generally fix them as paramount to the dimensional position of record; it may be presumed that a new description, with new dimensions, but reciting the monuments, will constitute a perpetuation of the original record, modified to represent the present facts.

Relocation surveys, or those made to establish boundaries of recorded existing parcels require the best skill, analytical faculty, and judgment of the land surveyor. He may take nothing for granted, but must verify every monument, mathematical computation, measurement, description, and legal interpretation. The final decision must be based on sound judgment of his analysis and a close adherence to the theory of majority probability. Inasmuch

as the surveyor has no judicial authority, he must do his work and submit his findings in the manner best calculated to maintain and support his opinion of the location in question.

The first step in relocation is to gather all information relating to the land to be surveyed—the deed description, title policy, description, tract or subdivision maps of the property and vicinity, descriptions and maps of adjoining properties, street center-line survey notes, and location and ties of existing monuments in the area if such ties are available.

The data thus assembled are to be platted, the existing dimensions computed for agreement and closure, adjoiner boundaries related to the instant property, and a preliminary paper location determined.

The plat so prepared is taken into the field; for reference, a connected survey is made, the surveyor discovering and tying all monuments and other topographic features into a single net. Back in the office, the field work and the record are correlated, monuments and descriptions are interpreted in mutual relationship, and a new plat is prepared for the final showing of the survey.

The second trip to the field (assuming that all necessary and discoverable data have been previously obtained) will set the monuments of the relocation; and these of course will conform with the plat and will bear any authorized markings or structure regulated by law.

The relocation of corners of sections, or of fractional parts of sections, is governed by the method and procedure as defined by the Bureau of Land Management (formerly, the General Land Office), U. S. Department of the Interior.[2] Very little deviation from the rules can be permitted; yet close study of the manual is necessary to arrive at the precise variant applicable to the case in hand.

Argument that monuments govern regardless of description or record is pertinent only if the original survey is precedent to the record. If the ownership is established, as in many cases, by a description drawn from a plat, or by compilation from record sources, then the monuments must be assumed as confirmatory rather than determinative and subject to valid replacement in conformity with interpretation of the documentary positioning. If the contrary view is held, it can be maintained only by support of legal decrees, or through modification by agreement of adjoiners properly recorded.

Monuments discovered in a survey cannot be accepted in themselves, as definitive of the corners they purport to occupy. It is required in every instance to identify them by a conformal record dimensional relation with some other physical object, to prove reasonably (1) that they are in the position of the original corners; (2) that in the event they are found as replacements of originals, they have not been deliberately or accidentally misplaced; and (3) that all persons concerned with their establishment and use, consent, actually or impliedly, to the accuracy of position.

Maps of groups of parcels or lots, when recited as a part of the description, must be considered as a whole in the location of any part. A reference to a

[2] "Manual of Surveying Instructions for the Survey of the Public Lands of the United States," U. S. Govt. Printing Office, Washington, D.C., 1947.

record in a document incorporates therein all the information and substance of such record applicable to the interpretation of the terms of such document. Consequently, the location of the corners of a lot in a block must be proportionately related to the corners of all other lots in the block. The surveyor must establish the block corners correctly, either from accepted street center lines, monuments shown on the map or existing physically, or from other sources. Having so fixed the exteriors, he sets his lot corners by the prorate method of measurement. Such a method is not rigid, however, because other control factors, for example, verifiable original lot corner positions, may force the abandonment of the prorate measurement. Here is where the evidence, theory, and judgment of the surveyor must be weighed and properly concluded.

Regardless of the fact that the basis of land ownership is the possession, visibly, by structural objects or boundary monuments of the land itself, a great percentage of present day holding and transfer of property is maintained through the record only; and the ground location is not identified by monumentation until the owner wishes to avail himself of its actual use. It follows then that, in most cases, the surveyor must predicate his work first on the record, and, second, on the possession.

Whenever the surveyor is called upon to locate by title of record, disregarding possessory claim, his procedure is governed almost entirely by the proper interpretation of the terms of the deed description. Except for specific physical identification of the tie and reference calls of the deed, the field examination reduces to a mere correlation of available information for purposes of analysis. In cases of this kind, the correct placement of a boundary from the record is superior to any occupational claim.

As far as the surveyor is concerned, title is the relation between the record claim of the proprietor and the actual ground position of such claim. It is his business to make that relation exact and correct. If, due to error in the record or holding, such conditions cannot be met under certain circumstances, he must inform his client of the facts, and withhold final location until proper adjustment is made, either by agreement or court action.

The questions of finance, relative value of property to survey, ethical distinctions, or efficiency of mechanical processes involved, are not considered in this paper. They have no bearing on the premise or conclusion that no survey location of land areas is complete unless it corresponds and coordinates with the record title or a proper modification of that title.

Summary

A broad view of the field of property boundary surveys and an analysis of the factors governing correct location of such boundaries indicate the following conditions necessary to successful operation:

(1) Skilled, experienced, and able professional men, trained in field, office, and title practices;

(2) Proper equipment for all necessary work;

(3) Complete information and data of record, together with any other available documentary material affecting the project at issue;

(4) Sufficient preliminary field work to supply corroborative physical information in connection with the project;

(5) Compilation, and sound, intelligent analysis of the title and legal requirements necessary to fix a proper ground location of title boundaries, or to provide a basis of adjustment through the medium of record agreement or court action; and

(6) Physical monumentation of boundaries when finally determined, together with recordation of any necessary maps and documents which will correctly delineate and describe such final boundaries—the physical location and the record title are thereby brought into agreement.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

PAPERS

CONTINUOUS ARCHES AND BENTS ANALYZED BY COLUMN ANALOGY

By T. M. Yu[1]

Synopsis

Basically, the method of column analogy, as introduced[2] by Hardy Cross, Hon. M. ASCE, in 1930, is applicable to beams, arches, and bents of not more than three redundant elements. This paper offers an extension of the method to continuous beams, arches, and bents, provided that there are no closed panels.

The procedure is built around four broad steps:

(*a*) Select one span (any span) at a time;
(*b*) Consider all parts adjacent to that span as its elastic foundations;
(*c*) Replace the elastic foundations (item (*b*)) by substitute members with characteristics such that the transformed span is equivalent to a span on elastic foundations; and
(*d*) Determine the elastic properties of substitute members and proceed with the application of the fundamental column analogy method.

The ideas presented, clarified by numerical examples, demonstrate that column analogy is a convenient tool for sketching influence lines. The writer concedes in advance that there may be better methods of structural analysis for specific cases but he offers the hope that, by excluding extensive discussion of such alternate methods, it may be possible to "sharpen" the column analogy "tool" for use in its particular fields of application.

Introduction

In its original form,[2,3] the method of column analogy introduced by Professor Cross is applicable only to single-span arches and bents with not more

Note.—Written comments are invited for immediate publication; to insure publication the last discussion should be submitted by **July 1, 1949.**

[1] Prof. of Civ. Eng., Chiao-Tung Univ., Shanghai, China.

[2] "The Column Analogy," by Hardy Cross, *Bulletin No. 215*, Univ. of Illinois Eng. Experiment Station, Urbana, Ill., 1930.

[3] "Continuous Frames of Reinforced Concrete," by Hardy Cross and N. D. Morgan, John Wiley & Sons, Inc., New York, N. Y., 1932.

than three redundant elements; and the analogy lies in a "* * * mathematical identity between the moments produced by continuity in a beam, bent, or arch and the fiber stresses in a short column eccentrically loaded."[4] In this paper the writer explores the possibility of extending the column analogy to the analysis of continuous arches or bents. It is requested that discussion be confined to an objective appraisal of this effort.

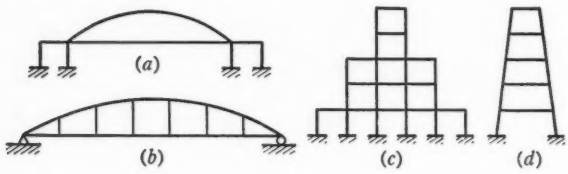


FIG. 1

The extension is not applicable to frames with closed panels (Fig. 1), and is therefore not intended to replace the moment distribution method. It is expected, however, to compare favorably with other methods in the determination of influence lines for continuous arches or bents.

Notation.—The letter symbols adopted for use in this paper are defined where necessary, in the text or by illustration, and are assembled alphabetically for convenience of reference in the Appendix.

Structural Formulas and Expressions.—The following expressions are used without further definition in this paper:

$$\text{Flexural rigidity} = E\,I$$

First moments of an elastic area are

$$A\,\bar{x} = \int \frac{x\,ds}{E\,I} \qquad (1a)$$

and

$$A\,\bar{y} = \int \frac{y\,ds}{E\,I} \qquad (1b)$$

Second moments of elastic area are

$$I_x = \int \frac{y^2\,ds}{E\,I} \qquad (2a)$$

$$I_y = \int \frac{x^2\,ds}{E\,I} \qquad (2b)$$

and

$$I_{xy} = \int \frac{x\,y\,ds}{E\,I} \qquad (2c)$$

[4] "Continuous Frames of Reinforced Concrete," by Hardy Cross and N. D. Morgan, John Wiley & Sons, Inc., New York, N. Y., 1932, p. 46.

Expressions for horizontal and vertical reactions, and for moments, due to "continuity," respectively, are: For symmetrical structures—

$$h_i = \frac{M_x}{I_x} \quad \ldots\ldots (3a)$$

$$v_i = \frac{M_y}{I_y} \quad \ldots\ldots (3b)$$

and

$$m_i = \frac{P}{A} + \frac{M_y}{I_y} x + \frac{M_x}{I_x} y \quad \ldots\ldots (3c)$$

and, for asymmetrical structures—

$$h_i = \frac{M'_x}{I'_x} \quad \ldots\ldots (4a)$$

$$v_i = \frac{M'_y}{I'_y} \quad \ldots\ldots (4b)$$

and

$$m_i = \frac{P}{A} + \frac{M'_y}{I'_y} x + \frac{M'_x}{I'_x} y \quad \ldots\ldots (4c)$$

in which

$$M'_y = M_y - \frac{I_{xy}}{I_x} M_x \quad \ldots\ldots (5a)$$

$$M'_x = M'_x - \frac{I_{xy}}{I_y} M_y \quad \ldots\ldots (5b)$$

$$I'_y = I_y - \frac{I^2_{xy}}{I_x} \quad \ldots\ldots (5c)$$

and

$$I'_x = I_x - \frac{I^2_{xy}}{I_y} \quad \ldots\ldots (5d)$$

The final values of total horizontal and vertical reactions, and moments are, respectively,

$$H = h_s - h_i \quad \ldots\ldots (6a)$$

$$V = v_s - v_i \quad \ldots\ldots (6b)$$

and

$$M = m_s - m_i \quad \ldots\ldots (6c)$$

Sign Convention.—According to Professor Cross,[5]

"* * * the bending moments in the column analogy are taken as positive when they produce tension on the bottom of a beam or on the inner side of an arch or bent. * * * Positive rotations are such as would be produced by positive bending moments. * * *

"For horizontal forces, pull at the abutments (tension in girder of a bent) is positive.

"For vertical forces, use the customary convention for shear in girders. (Positive shear is up on the left, down on the right.)

[5] "Continuous Frames of Reinforced Concrete," by Hardy Cross and N. D. Morgan, John Wiley & Sons, Inc., New York, N. Y., 1932, pp. 55–56.

"The correct signs for changes in reaction are then automatically determined by the column analogy, and the correct values for the reactions follow algebraically from $V = v_s - v_i$; $H = h_s - h_i$.

"Positive displacements correspond to positive forces. Positive bending moment is positive force times positive distance (distance measured up, or to the right, from the force to the section)."

The Substitute Member

The beam, arch, or bent AB in Fig. 2(*a*) is shown in Fig. 2(*b*) supported at the right end by the substitute member BB′. The characteristics of this method are such that the two structures shown in Fig. 2 are equivalent as far

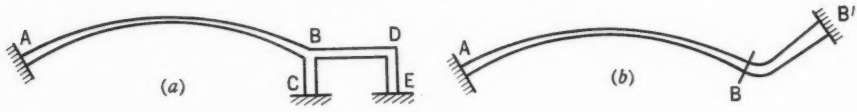


Fig. 2

as redundant elements (and therefore deflections) in span AB are concerned, provided that external loads act on span AB only. (In Fig. 2(*a*) if column BC is to be analyzed, instead of span AB, it is assumed to be supported at point B by the "elastic foundation" ABDE. Similarly, the beam-and-column span BDE is assumed to be supported at point B by the "elastic foundation" ABC.)

Elastic Properties of the Substitute Member

In order that the substitute member BB′, Fig. 2(*b*), shall be equivalent to the elastic foundation BCDE (Fig. 2(*a*)), it is necessary (and sufficient) that the original members and the substitute member give the same deflections and rotations at point B when equal loads are applied at point B. Member AB is temporarily assumed to be cut at point B, making the elastic foundation BCDE (or the substitute member BB′) an independent member. This principle is illustrated in Fig. 3.

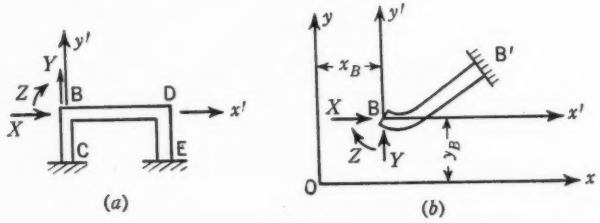


Fig. 3

The deflections of the elastic foundation BCDE (Fig. 3(*a*)) at joint B caused by unit influence loads at that point can be readily computed, making use of the column analogy or other methods. Similarly, it is possible to write the corresponding deflections for the substitute member BB′. These deflections are given in the most general form, in Table 1. Equating the corresponding deflections, lines 9 and 8, Table 1, yield, respectively,

TABLE 1.—FORMULAS FOR A SUBSTITUTE MEMBER AT POINT B, FIG. 3

Load at B, Fig. 3	Line	Deflection at B	Elastic foundation (BCDE)	Substitute member (BB′)
$X=1$ $Y=Z=0$	1	Horizontal	δ_{xx}	$\int \frac{(y')^2\,ds}{E\,I}$
	2	Vertical	δ_{xy}	$-\int \frac{x'\,y'\,ds}{E\,I}$
	3	Angular	δ_{xz}	$-\int \frac{y'\,ds}{E\,I}$
$Y=1$ $X=Z=0$	4	Horizontal	$\delta_{yx}=\delta_{xy}$	$-\int \frac{x'\,y'\,ds}{E\,I}$
	5	Vertical	δ_{yy}	$\int \frac{(x')^2\,ds}{E\,I}$
	6	Angular	δ_{yz}	$\int \frac{x'\,ds}{E\,I}$
$Z=1$ $X=Y=0$	7	Horizontal	$\delta_{zx}=\delta_{xz}$	$-\int \frac{y'\,ds}{E\,I}$
	8	Vertical	$\delta_{zy}=\delta_{yz}$	$\int \frac{x'\,ds}{E\,I}$
	9	Angular	δ_{zz}	$\int \frac{ds}{E\,I}$

$$\int \frac{ds}{E\,I} = \delta_{zz} \qquad (7a)$$

and

$$\int \frac{x'\,ds}{E\,I} = \delta_{yz} \qquad (7b)$$

in which ds is an elementary segment of the substitute member, BB′, whose coordinates are x',y', referred to joint B as the origin, and whose flexural rigidity is $E\,I$ (E is the modulus of elasticity; and I is the moment of inertia of the cross section). Dividing Eq. 7*b* by Eq. 7*a*,

$$\bar{x}' = \frac{\delta_{yz}}{\delta_{zz}} \qquad (8a)$$

By a similar logic, dividing line 7, Table 1, by line 9,

$$\bar{y}' = -\frac{\delta_{xz}}{\delta_{zz}} \qquad (8b)$$

Eqs. 8*a* and 8*b* give the coordinates of the centroid of the elastic area of the substitute member.

The second moments referred to the elastic center of BB′ $\left(\text{or the centroid of } \int \frac{ds}{E\,I}\right)$ are

$$I_{gy} = \int \frac{(x')^2\,ds}{E\,I} - (\bar{x}')^2 \int \frac{ds}{E\,I} = \delta_{yy} - \frac{\delta^2_{yz}}{\delta_{zz}} \qquad (9a)$$

$$I_{gx} = \int \frac{(y')^2\,ds}{E\,I} - (\bar{y}')^2 \int \frac{ds}{E\,I} = \delta_{xx} - \frac{\delta^2_{xz}}{\delta_{zz}} \qquad (9b)$$

and

$$I_{oxy} = \int \frac{x'\,y'\,ds}{E\,I} - \bar{x}'\,\bar{y}' \int \frac{ds}{E\,I} = -\,\delta_{xy} + \frac{\delta_{xz}\,\delta_{yz}}{\delta_{zz}} \quad \ldots\ldots\ldots (9c)$$

Translating the axes from point B to the new origin O, Fig. 3(*b*),

$$\bar{x} = x_B + \bar{x}' \quad \ldots\ldots\ldots (10a)$$

and

$$\bar{y} = y_B + \bar{y}' \quad \ldots\ldots\ldots (10b)$$

from which

$$I_y = \delta_{yy} - \frac{\delta^2_{yz}}{\delta_{zz}} + (x_B + \bar{x}')^2\,\delta_{zz} \quad \ldots\ldots\ldots (11a)$$

$$I_x = \delta_{xx} - \frac{\delta^2_{xz}}{\delta_{zz}} + (y_B + \bar{y}')^2\,\delta_{zz} \quad \ldots\ldots\ldots (11b)$$

and

$$I_{xy} = -\,\delta_{xy} + \frac{\delta_{xz}\,\delta_{yz}}{\delta_{zz}} + (x_B + \bar{x}')\,(y_B + \bar{y}')\,\delta_{zz} \quad \ldots\ldots\ldots (11c)$$

In using Eqs. 9 to 11, positive unit loads or moments at point B act in the directions or sense shown in Fig. 3. The same directions or sense is assumed positive for deflections or rotation, respectively. Then the final results will automatically give the correct algebraic signs, regardless of the side of the beam, arch, or bent on which the elastic foundation lies.

It is not necessary (nor is it possible) to know the flexural rigidity $E\,I$ at any particular section of the substitute member; it is not necessary (nor is it possible) to know even the location of its axis. With the properties determined by the formulas in Table 1, and by Eqs. 9 to 11, the designer can proceed with the application of the column analogy.

Example 1

The frame shown in Fig. 4 is solved in Tables 2 and 3, which are self-explanatory to anyone who has a proper understanding of the column analogy method. The tabulation and the notations follow those of Professor Cross[2,3] except that I_x, I_y, M_x, and M_y represent, respectively, I_y, I_x, M_y, and M_x in the Cross notation. These changes are made to be consistent with those commonly used in textbooks on engineering mechanics.

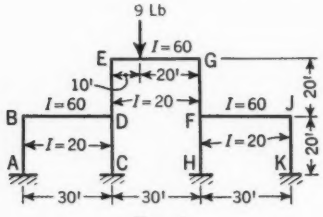


Fig. 4

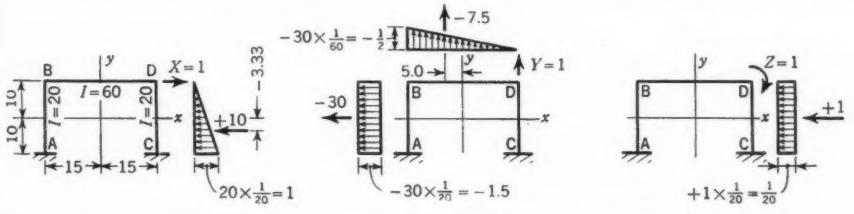
Tables 2 constitute the complete analysis of moments, by column analogy, in the elastic foundation ABDC (which is the same as HFJK) when unit influence loads are applied at joint D, Fig. 4. It can be noted by inspection of Table 2*B* that the analysis of moments due to the unit influence load $Y = 1$ is unnecessary; but this has been included to make the example reasonably complete. The deflections at joint D, namely, δ_{xx}, δ_{xy}, δ_{xz}, δ_{yy}, and

δ_{zz} are then computed (Table 2C(*b*)). It is convenient to note that $\delta_{xz} = \delta_{zx} = +0.38$ gives a check to the correctness of the analysis. Finally, the properties of the substitute member DD′ are computed by formulas in Table 1 and

TABLE 2A.—PROPERTIES OF BENT ABDC, EXAMPLE 1

No.	Bar (see Fig. 4)	Given Data				Computed Properties					
		L	I	x	y	Elastic Area	First Moments		Second Moments		
						a	$a\,x$	$a\,y$	$a\,x^2 + i_y$	$a\,y^2 + i_x$	$a\,x\,y + i_{xy}$
1	AB	20	20	−15	0	1.0	−15	0	225	33.33	0
2	BD	30	60	0	+10	0.5	0	+5	37.5	50.00	0
3	DC	20	20	+15	0	1.0	+15	0	225	33.33	0
4	Correction to elastic centroid					....	....	....	0	−10.0	0
5	Σ	....	....	$\frac{0}{2.5} = 0 = \bar{x}$	$\frac{+5}{2.5} = +2 = \bar{y}$	2.5	0	+5	487.5 $= I_x$	106.7 $= I_y$	0 $= I_{xy}$

TABLE 2B.—ELASTIC LOADS AND FORMULAS FOR m_i, BENT ABDC, EXAMPLE 1



No.	Bar (see Fig. 4)	Horizontal Force $X = 1$					Vertical Force $Y = 1$					Moment $Z = 1$				
		Given			Computed		Given			Computed		Given			Computed	
		P	x	y	M_y	M_x	P	x	y	M_y	M_x	P	x	y	M_y	M_x
1	AB	....	....	....	....	....	−30.0	−15	0	+450	0	..	....	..	....	....
2	BD	....	....	....	....	....	− 7.5	− 5	+10	+ 37.5	−75	..	....	..	....	....
3	DC	+10	−15	−3.33	+150	−33.3	....	....	....	....	....	1	+15	0	+15	0
4	Corrected[a]	....	....	....	0	−20.0	....	....	....	0	+75	..	....	..	0	−2
5	Σ	+10	....	....	+150	−53.3	−37.5	....	....	+487.5	0	1	....	..	+15	−2
6	m_i	$= \frac{10}{2.5} + \frac{150}{487.5}x - \frac{53.3}{106.7}y$					$= -\frac{37.5}{2.5} + \frac{487.5}{487.5}x + 0$					$= \frac{1}{2.5} + \frac{15}{487.5}x - \frac{2}{106.7}y$				

[a] Correction to elastic centroid.

TABLE 2*C*.—Computation of Moments and Deflections, Bent ABDC, Example 1

No.	Point (see Fig. 4)	x	y	Horizontal Force $X = 1$			Vertical Force $Y = 1$			Moment $Z = 1$		
				m_s	m_i	M	m_s	m_i	M	m_s	m_i	M
				(*a*) Moments, $M = m_s - m_i$								
1	A	−15	−12	0	+ 5.38	−5.38	−30	−30	0	0	+0.163	−0.163
2	B	−15	+ 8	0	− 4.62	+4.62	−30	−30	0	0	−0.212	+0.212
3	D_{DB}	+15	+ 8	0	+ 4.62	−4.62	0	0	0	0	+0.712	−0.712
4	D_{DC}	+15	+ 8	0	+ 4.62	−4.62	0	0	0	+1	+0.712	+0.288
5	C	+15	−12	+20	+14.62	+5.38	0	0	0	+1	+1.087	−0.087
				(*b*) Deflections at Joint D, Fig. 4								
6	Horizontal			$\delta_{xx} = +20.5$			$\delta_{yx} = 0$			$\delta_{xx} = +0.38$		
7	Vertical			$\delta_{xy} = 0$			$\delta_{yy} = 0$			$\delta_{xs} = 0$		
8	Rotation			$\delta_{xs} = +0.38$			$\delta_{ys} = 0$			$\delta_{ss} = +0.1005$		

TABLE 3*A*.—Elastic Load and Formula for m_i, Bent DEGF, Example 1 (Figs. 5 and 6)

Given Data					Computed Properties						Elastic Load				
Bar	L	I	x	y	Area[a]	First Moments		Second Moments			P	x	y	$M_y = Px$	$M_x = Py$
					a	ax	ay	$ax^2 + i_y$	$ay^2 + i_x$	$axy + i_{xy}$					
DE	20	20	−15	0	1.000	−15.00	0	225.0	33.33	0	....	....	....	....	....
EG	30	60	0	+10	0.500	0	+5.00	37.5	50.00	0	+15.0	−1.67	+10.0	−25.0	+150
GF	20	20	+15	0	1.000	+15.00	0	225.0	33.33	0	....	....	....	....	....
DD′[b]	....	....	−15	....	0.1005	− 1.51	−1.39	22.6 0	19.1 19.1	+9.4 0	....	....	....	....	....
FF′[b]	....	....	+15	....	0.1005	+ 1.51	−1.39	22.6 0	19.1 19.1	−9.4 0	....	....	....	....	....
....	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....
Correction to centroidal axis					....	....	....	0	−1.8	0	....	....	....	0	−12.3
Σ	....	....	0[c]	0.822[c]	2.701	0	+2.22	532.7[d]	191.3[d]	0	+15.0	....	....	−25.0[e]	+137.7[e]

$$m_i = \frac{15}{2.701} - \frac{25}{532.7}x + \frac{137.7}{191.3}y$$

[a] Elastic area. [b] See text (Example 1). [c] $\bar{x}$ and $\bar{y}$, respectively. [d] I_y and I_x, respectively. [e] M_y and M_x, respectively.

TABLE 3*B*.—Moments $M = m_s - m_i$, Bent DEGF, Example 1 (Figs. 5 and 6)

No.	Point	x	y	m_s	m_i	M (see Fig. 5(*b*))
1	D	−15	−10.82	0	− 1.52	+ 1.52
2	E	−15	+ 9.18	0	+12.86	−12.86
3	G	+15	+ 9.18	0	+11.44	−11.44
4	F	+15	−10.82	0	− 2.94	+ 2.94

by Eqs. 7 to 9, as follows:

$$\int \frac{ds}{I} = \delta_{zz} = +0.1005$$

$$\bar{x} = \frac{\delta_{yz}}{\delta_{zz}} = 0$$

$$\bar{y} = -\frac{\delta_{xz}}{\delta_{zz}} = -3.78$$

$$I_{gy} = \delta_{yy} - \frac{\delta^2_{yz}}{\delta_{zz}} = 0$$

$$I_{gx} = \delta_{xx} - \frac{\delta^2_{xz}}{\delta_{zz}} = 19.1$$

$$I_{gxy} = -\delta_{xy} + \frac{\delta_{xz}\,\delta_{yz}}{\delta_{zz}} = 0$$

Similarly to Tables 2, Tables 3 contain the complete analysis, by column analogy, of the moments in bay DEGF, Fig. 4, considered to be supported by DD′ and FF′. The moments in bents ABDC and HFJK are not evaluated.

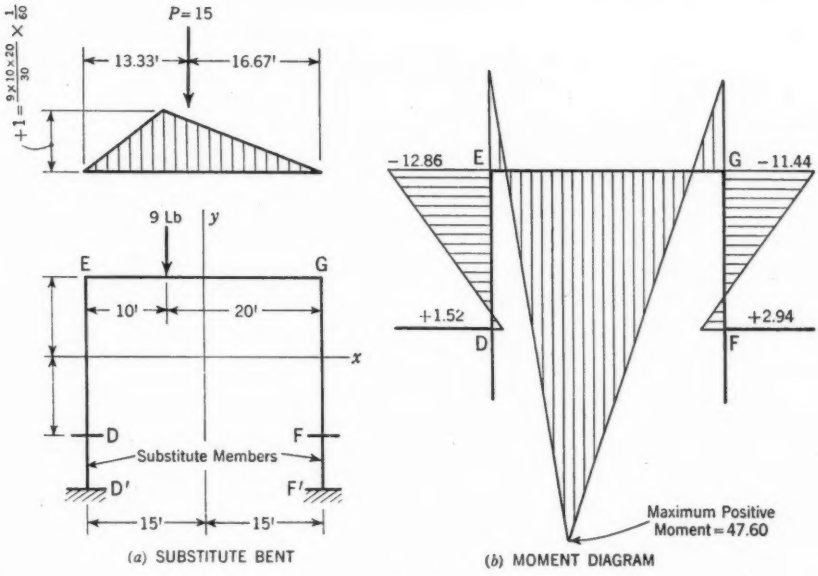


FIG. 5

If desired, they can be readily computed by finding the moment and shear transmitted by column DE at point D (or by column FG at point F), Fig. 5, and making subsequent computations by superposition (using the results in Table 2*C*, with proper attention to signs).

Sketching Influence Lines; Example 2

The determination of influence lines by the principle of virtual work has been treated elsewhere,[6] and the application of the column analogy to the same problem is also well known.[2,3] A simple example will be given to illustrate the usefulness of the method.

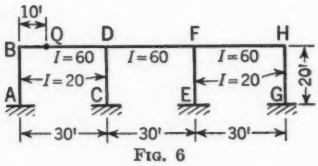


Fig. 6

A three-bay continuous bent is analyzed for bending moment, shear, and axial tension at point Q, Fig. 6. The analysis begins with the determination of the properties of the substitute member DD′ (see Tables 4) and the moments due to virtual displacements at point Q, Fig. 6 (see Tables 5).

In Table 4*C*(*b*), $\delta_{xx} = 0.317$ and $\delta_{zz} = 0.312$, indicating an error as the result of slide rule inaccuracies. In subsequent computations the designer would use the average of these two values. Finally, the properties of the substitute member DD′ are computed by formulas in Table 1 and by Eqs. 7 to 9, as follows:

$$\int \frac{ds}{E\,I} = \delta_{zz} = +\,0.097$$

$$\bar{x} = \frac{\delta_{yz}}{\delta_{zz}} = 0$$

$$\bar{y} = -\,\frac{\delta_{xz}}{\delta_{zz}} = 0$$

$$I_{qy} = 0$$

$$I_{qx} = \delta_{xx} - \frac{\delta^2_{xz}}{\delta_{zz}} = 12.58$$

$$I_{qxy} = 0$$

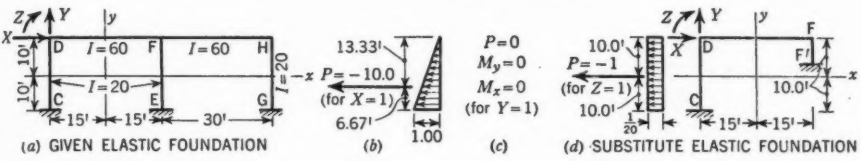
TABLE 4*A*.—Properties of Bent CDFF′, Example 2 (Fig. 6)

Given Data						Computed Properties					
No.	Bar[a]	*L*	*I*	*x*	*y*	Elastic Area	First Moments		Second Moments		
						a	*a x*	*a y*	$a\,x^2 + i_y$	$a\,y^2 + i_x$	$a\,x\,y + i_{xy}$
1	CD	20	20	−15	0	1.00	−15	0	225.0	33.3	0
2	DF	30	60	0	+10	0.50	0	+5.0	37.5	50.0	0
3	FF′	....	....	+15	+ 6.22	0.1005	+ 1.51	+0.63	22.6	{3.9 / 19.1}	+ 9.4
4	Second moments corrected to elastic centroid							....	−113.7	−19.8	+47.5
5	Σ	....	....	−8.43[b]	+3.52[b]	1.60 = *A*	−13.49	+5.63	171.4 = I_y	86.5 = I_z	+56.9 = I_{zy}
6	Corrected for dissymmetry					....	....	....	134.0 = I'_y	67.6 = I'_z	....

[a] See Fig. 6 and Table 4*B*. For the properties of the substitute member FF′, see lines 4 and 5, Table 3*A*, and the text under Example 1. [b] Values of $\bar{x}$ and $\bar{y}$, respectively.

[6] "Structural Theory," by Hale Sutherland and H. L. Bowman, John Wiley & Sons, Inc., New York, N. Y., 3d Ed., 1942.

TABLE 4*B*.—Elastic Loads and Formulas for m_i, Bent CDFF′, Example 2



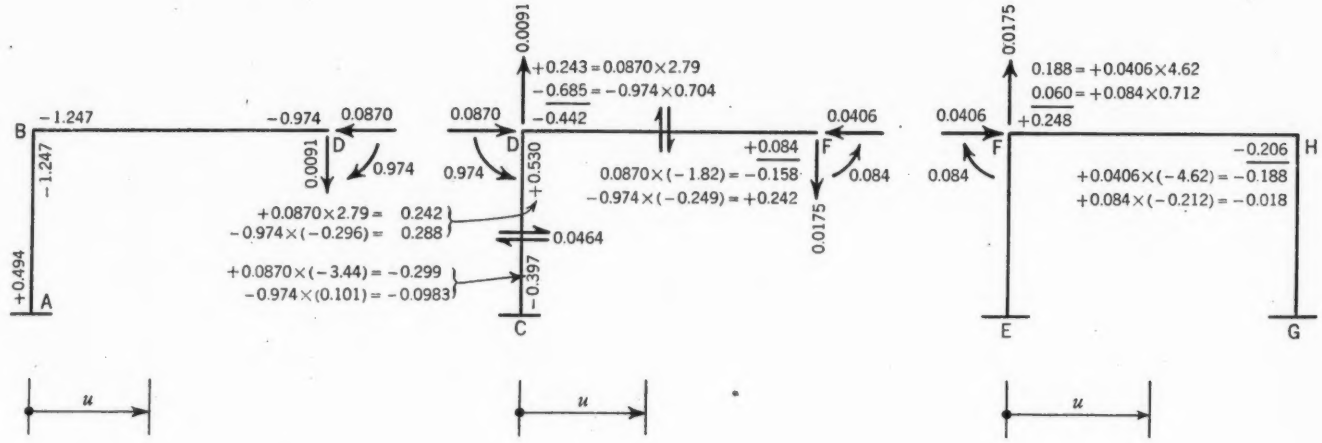
(*a*) GIVEN ELASTIC FOUNDATION (*b*) (*c*) (*d*) SUBSTITUTE ELASTIC FOUNDATION

No.	Bar (see Fig. 6)	Horizontal Force $X = 1$					Vertical force $Y = 1$	Moment $Z = 1$				
		Given			Computed			Given			Computed	
		P	x	y	M_y	M_x		P	x	y	M_y	M_x
1	CD	−10.0	−15	−3.33	+150	+33.3		−1.00	−15	0	15	0
2	DF	....	....	....	....	....	$P=0$	....	....	..	....	....
3	FF′	....	....	....	....	....	$M_y=0$	....	....	..	....	....
4	Correction[a]	....	....	....	−84.3	+35.2	$M_x=0$	....	....	..	−8.43	+3.52
5	Σ	−10.0	....	....	65.7	+68.5		−1.00	....	..	$+6.57 = M_y$	$+3.52 = M_x$
6	Corrected for dissymmetry				+20.6	+46.7		....	....	..	$+4.25 = M'_y$	$1.34 = M'_x$
7	m_i	$= -\frac{10}{1.6} + \frac{20.6}{134}x + \frac{46.7}{67.6}y$					$= 0$	$= -\frac{1}{1.6} + \frac{4.25}{134}x + \frac{1.34}{67.6}y$				

[a] Correction to elastic centroid.

TABLE 4*C*.—Computation of Moments and Deflections, Bent CDFF′, Example 2

No.	Point (see Fig. 6)	x	y	Horizontal Force $X = 1$			$Y = 1$			Moment $Z = 1$		
				m_s	m_i	M	m_s	m_i	M	m_s	m_i	M
				(*a*) Moments, $M = m_s - m_i$								
1	C	− 6.57	−13.52	−20	−16.56	−3.44	0	0	0	−1.00	−1.101	+0.101
2	Dcd	− 6.57	+ 6.48	0	− 2.79	+2.79	0	0	0	−1.00	−0.704	−0.296
3	Ddf	− 6.57	+ 6.48	0	− 2.79	+2.79	0	0	0	0	−0.704	+0.704
4	F	+23.43	+ 6.48	0	+ 1.82	−1.82	0	0	0	0	+0.249	−0.249
				(*b*) Deflections at Joint D, Fig. 6								
7	Horizontal			$\delta_{xx} = +13.6$			$\delta_{yx} = 0$			$\delta_{xx} = +0.312$		
8	Vertical			$\delta_{xy} = 0$			$\delta_{yy} = 0$			$\delta_{xy} = 0$		
9	Angular			$\delta_{xz} = +0.317$			$\delta_{yz} = 0$			$\delta_{xz} = +0.097$		



$0 \lesssim u \lesssim 10.0 \quad \Delta_s = 0.667u$

$10 \lesssim u \lesssim 30.0 \quad \Delta_s = 10 - 0.333u$

$\Delta_i = -0.289u + 0.01039u^2 - 0.0000253u^3$

$\theta_{DB} = 0.067$ ↻

$\Delta_s = 0$

$\Delta_i = -0.0667u + 0.00368u^2 - 0.0000487u^3$

$\theta_{DF} = 0.067$ ↻ $\quad \theta_{FD} = 0.024$ ↺

$\Delta_s = 0$

$\Delta_i = 0.0242u - 0.00206u^2 + 0.0000420u^3$

$\theta_{FH} = 0.024$ ↺

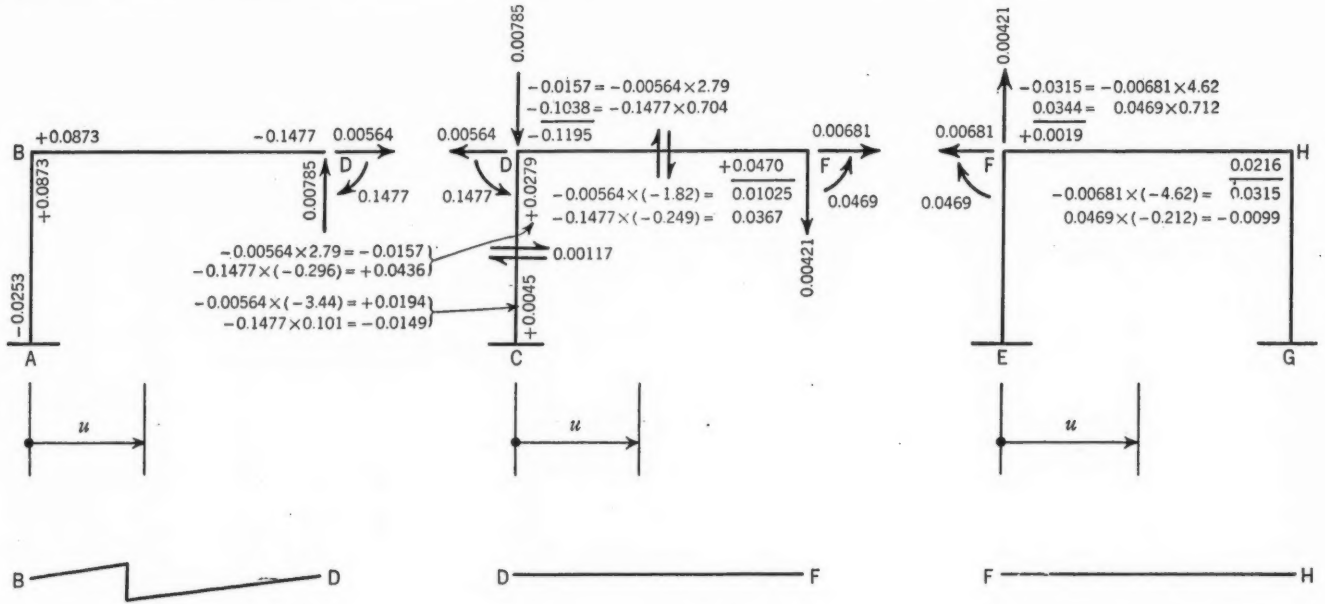


FIG. 7.—FORMULAS FOR VIRTUAL DISPLACEMENTS, EXAMPLE 2

(*Continued on next page*)

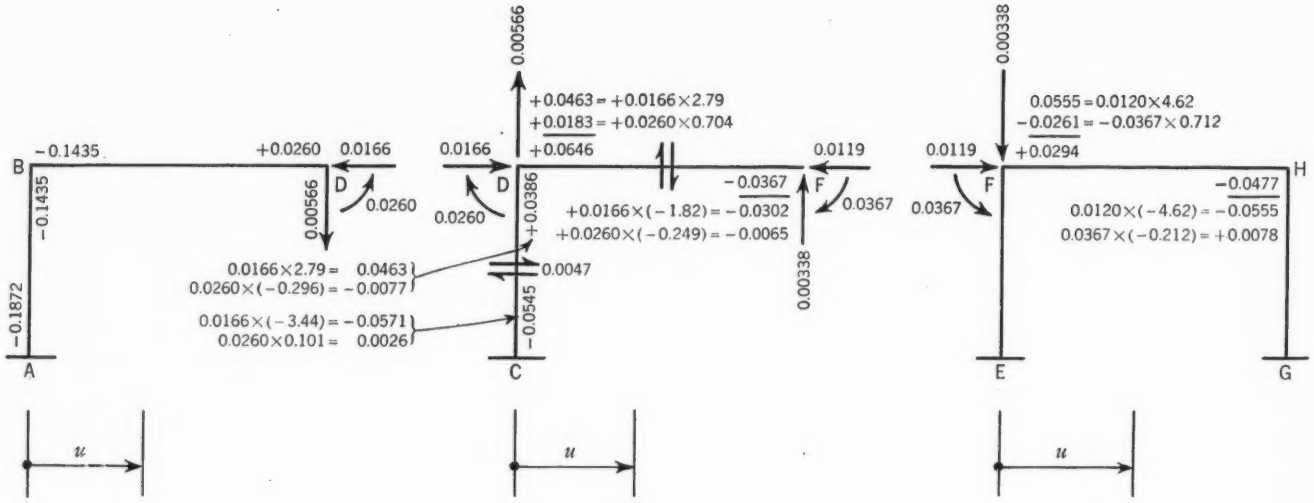
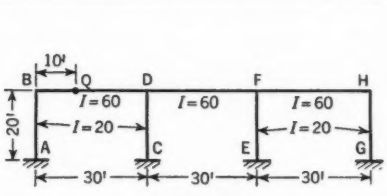
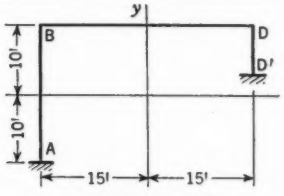


FIG. 7.—*(Continued)*

TABLE 5A.—Properties of Bent ABDD′ (For Computation of Moments Due to Virtual Displacements); Example 2



GIVEN BENT

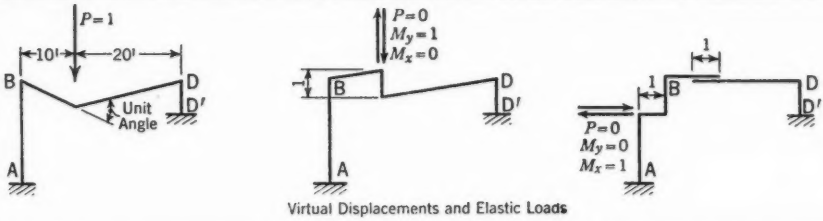


SUBSTITUTE BENT

No.	Given Data: Bar[a] (see Fig. 6)	L	I	x	y	Computed Properties: Elastic Area a	First Moments a x	a y	Second Moments a x² + i_y	a y² + i_x	a x y + i_{xy}
1	AB	20	20	−15	0	1.000	−15.00	0	225.0	33.3	0
2	BD	30	60	0	+10	0.500	0	+5.000	37.5	50.0	0
3	DD′	..	..	+15	+ 6.75	0.097	+ 1.45	+0.655	21.8	4.4	+ 9.8
4										12.6	
5	Second moment correction to centroid					....	....	....	−114.9	−20.1	+48.0
6	Σ	..	..	− 8.49[b]	+ 3.55[b]	1.597	−13.55	5.655	169.4 = I_y	80.2 = I_x	+57.8 = I_{xy}
7	Corrected for dissymmetry				....	....	....	....	127.8 = I'_y	60.5 = I'_x	....

[a] For properties of substitute member DD′, see text under Example 2. [b] $\bar{x}$ and $\bar{y}$, respectively.

TABLE 5B.—Elastic Loads on Bent ABDD′ and Moments Due to Virtual Displacements; Example 2



Virtual Displacements and Elastic Loads

No.	Bar[a] (see Fig. 6)	P = 1: Given P	x	y	Computed M_y	M_x	M_y = 1: Given P	x	y	Computed M_y	M_x	M_x = 1: Given P	x	y	Computed M_y	M_x
1	AB	....	....	....	....	....	..	..	..	....	....	..	..	..	....	+1.00
2	BD	+1.0	−5	+10	−5.0	+10.0	0	..	..	+1.00	0	..	..	..	....	....
3	DD′	....	....	....	....	....	..	..	..	....	....	..	..	..	....	....
4	Correction[b]	....	....	....	+8.49	−3.55	..	..	..	0	0	..	..	..	0	0
5	Σ	1.0	....	....	+3.49[c]	+ 6.45[c]	0	..	..	+1.00[c]	0[c]	0	..	..	0[c]	+1.00[c]
6	Corrected for dissymmetry				−1.16[d]	+ 5.26[d]	..	..	..	+1.00[d]	−0.341[d]	..	..	..	−0.721[d]	+1.00[d]
7	m_i	$= \frac{1}{1.597} - \frac{1.16}{127.8}x + \frac{5.26}{60.5}y$					$= \frac{1}{127.8}x - \frac{0.341}{60.5}y$					$= -\frac{0.721}{127.8}x + \frac{1}{60.5}y$				

[a] For properties of substitute member DD′, see text under Example 2. [b] Correction to elastic centroid. [c] These are values of ΣM_y and ΣM_x, respectively. [d] These are values of M'_y and M'_x, respectively.

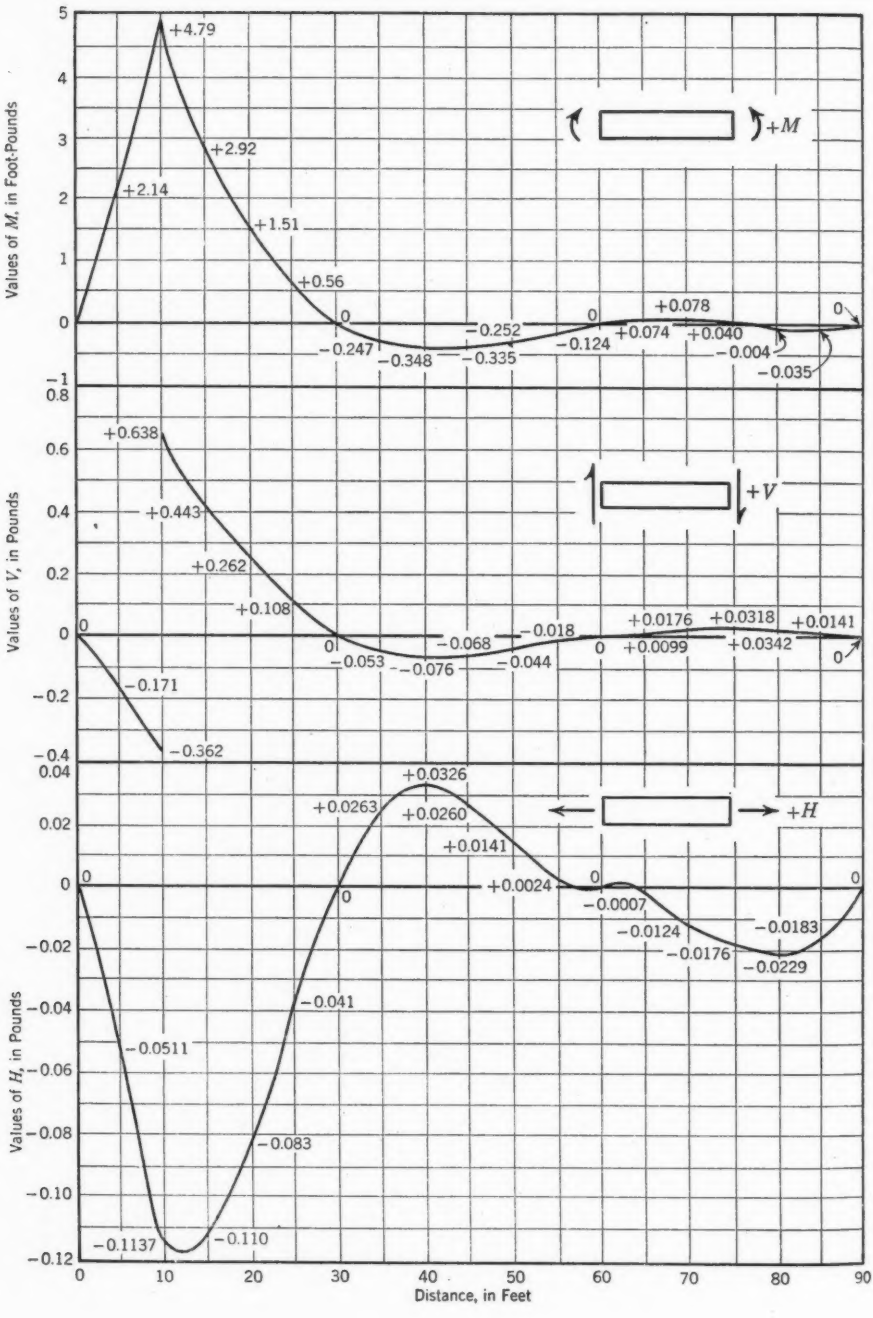


FIG. 8.—INFLUENCE ORDINATES FOR MOMENT, SHEAR, AND AXIAL TENSION AT *Q*, EXAMPLE 2

TABLE 5C.—COMPUTATION OF REACTIONS AND MOMENTS DUE TO VIRTUAL DISPLACEMENTS, BENT ABDD′, EXAMPLE 2

No.	Point (see Fig. 6)	x	y	$P = 1$			$M_y = 1$			$M_x = 1$		
				m_s	m_i	M	m_s	m_i	M	m_s	m_i	M
				MOMENTS $M = m_s - m_i$								
1	A	− 6.51	−13.55	0	−0.494	+0.494	0	+0.0253	−0.0253	0	−0.1872	+0.1872
2	B	− 6.51	+ 6.45	0	+1.247	−1.247	0	−0.0873	+0.0873	0	+0.1435	−0.1435
3	D	23.49	+ 6.45	0	+0.974	−0.974	0	+0.1477	−0.1477	0	−0.0260	+0.0260

Fig. 7 gives the moments of the three spans. The moments in girders DF and FH, which belong to the elastic foundation, are found by superposition, making use of the results of Tables 4 and Tables 2. The ordinates of the influence lines, represented by

$$y = \Delta_s + \Delta_i \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12)$$

are then computed and given in Fig. 8, in which Δ_s is the static deflection, and Δ_i is the deflection due to continuity.

CONCLUSIONS

In this paper the writer has attempted to do for the column analogy what T. Y. Lin,[7] Assoc. M. ASCE, J. A. Wise,[8] and others did for the method of moment distribution. Each of these efforts, including the present paper, can be properly criticized on the ground that apparent savings of time in computations are effected at the expense of preliminary work. Nevertheless, the ideas offered for discussion in this paper seem to possess the following merits: By avoiding simultaneous equations, computation work can be done more quickly and accurately; and, as a consequence, it is especially useful as a device for reviewing designed structures.

APPENDIX. NOTATION

The letter symbols used in this paper conform essentially with those adopted by Professor Cross in his exposition of the column analogy,[2,3] and also with tentative American Standard Letter Symbols for Structural Analysis (ASA—ZIOH—1947).

A = total elastic area = $\int \frac{ds}{EI}$;
a = elastic area of a member;
E = modulus of elasticity;

[7] "A Direct Method of Moment Distribution," by T. Y. Lin, *Transactions*, ASCE, Vol. 102, 1937, p. 561.

[8] "A Precise Moment Distribution Method," by Joseph A. Wise, *Journal*, A.C.I., November, 1938, p. 93.

FIG. 8.—INFLUENCE ORDINATES FOR MOMENT, SHEAR, AND AXIAL TENSION AT Q, EXAMPLE 2

H = total horizontal reaction of the structure (Eq. 6*a*);

h = horizontal reaction of a structure, with subscripts s and i denoting, respectively, "statically determinate structure" and "due to continuity";

I = second moment of the area of the cross section of a member ($E\,I$ = flexural rigidity);

I_x, I_y, and I_{xy} = the second moments of the elastic area, about axes x, y, and xy, respectively; for asymmetric members only, I', with appropriate subscript, denotes the second moment corrected for dissymmetry;

i = second moment of elastic area for a single member only, with subscripts corresponding to I;

M = total bending moment at any point of a structure (Eq. 6*c*);

M_x and M_y = the total static moments of the elastic load about axes x and y, respectively; for asymmetric members only, M', with appropriate subscript, denotes the total static moments of the elastic load, corrected for dissymmetry;

m = bending moment at any point of a structure, with subscripts s and i, respectively, denoting "statically determinate structure" and "due to continuity";

P = elastic load = $\int \frac{m_s\,ds}{E\,I}$;

s = length; ds denotes any elementary segment of any member or substitute member, whose coordinates are x and y, referred to a given point as origin, and whose flexural rigidity is $E\,I$;

V = total vertical reaction of a structure (Eq. 6*b*);

v = vertical reaction of a structure, with subscripts s and i denoting, respectively, "statically indeterminate structure" and "due to continuity";

X = horizontal influence load;

x = distance parallel to the x-axis; x', or x with appropriate subscript, refers to some other origin; $\bar{x}$ denotes a centroidal value of x;

Y = vertical influence load;

y = distance parallel to the y-axis; y', or y with appropriate subscript, refers to some other point as origin; $\bar{y}$ denotes a centroidal value of y;

Z = influence torque;

Δ = static linear deflection, with subscripts s and i denoting "in statically indeterminate structure" and "due to continuity," respectively;

δ = deflection at the supporting point of the "elastic foundation" due to unit influence load or torque acting at the same point; subscripts indicate the directions of the influence load and of the deflections, thus δ_{xy} denotes the y-component of the linear deflection due to influence load X = unity; and

θ = static rotary deflection in the structure.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

PAPERS

ANALYSIS OF PILE FOUNDATIONS WITH BATTER PILES

By A. Hrennikoff,[1] Assoc. M. ASCE

Synopsis

This paper presents a method of analysis of pile loads in foundations involving nonparallel piles, when both longitudinal and lateral resistances of piles are taken into account. Although the basic ideas underlying this method are substantially the same as in the other similar methods, the manner of applying these ideas is different in so far as a clear separation is made of the aspects of the problem that are reasonably definite and generally accepted from those that are uncertain. The element of uncertainty is present in the form of three coefficients, characterizing deformability of the piles, whose values can only be roughly estimated. By way of approximation, the two less important of these coefficients may be ignored and the third may be expressed in terms of data possessed by the designer, which lead to workable formulas suitable for practical use. A method of estimating the pile coefficients is also given, and by varying the values of these coefficients over a wide range, it is shown by an example that the error introduced by ignoring the minor coefficients is only moderate.

Pile Foundations

Batter piles are not used in foundations of structures carrying horizontal loads as often as their advantages would justify. Perhaps one of the reasons for this anomaly is the absence of a workable method of analysis.

In this connection one cannot fail to notice a conspicuous inconsistency in the usual design practice. On the one hand, many foundations carrying horizontal loads are provided with only vertical piles, so that all the horizontal restraint is caused by the lateral pile resistances alone. On the other hand, when the arrangement is such that two or more groups of nonparallel piles are present, the lateral pile resistances are completely ignored, as if they were of

Note.—Written comments are invited for immediate publication; to insure publication the last discussion should be submitted by **July 1, 1949.**

[1] Prof., Civ. Eng., Univ. of British Columbia, Vancouver, B. C., Canada.

no consequence. This approach is out of line with the one adopted in the case of parallel piles.

It may be pointed out that pile foundations very seldom, if ever, fail because the piles break, but almost invariably fail because the movement of the foundation becomes excessive. This statement is in full agreement with the recognized method of determining the allowable loads on piles both in longitudinal and in transverse directions by loading tests. Thus, most building codes specify that the safe axial load on a pile must be such that the settlement produced does not exceed a definite small value, such as $\frac{3}{8}$ in. With regard to the allowable lateral loads most of the codes are silent, but here again resistances specified by some authorities are based on definite small displacements of the pile heads.[2]

In the foregoing it was shown that, from the viewpoint of design, the pile load is considered merely as a symbol for deflection. If a specification limits the pile load to 20 tons, it is not because a 25-ton load is believed to break it, or at least bring it too close to failure, but because the settlement corresponding to it is expected to be excessive.

This symbolic use of the allowable load is not limited to the piles alone, but is common in other cases as well. For example, the allowable load on soil is governed largely by the expected settlement of the structure.

The close relation existing between the allowable load on a pile and the resultant displacement has an important bearing on the validity of the generally accepted method of pile analysis, previously referred to, which consists of disregarding the lateral pile resistances. One might think that, once the lateral pile resistance is omitted, the total pile resistance is underestimated, and the design is conservative. This conclusion, however, generally is not correct, as is shown by the following.

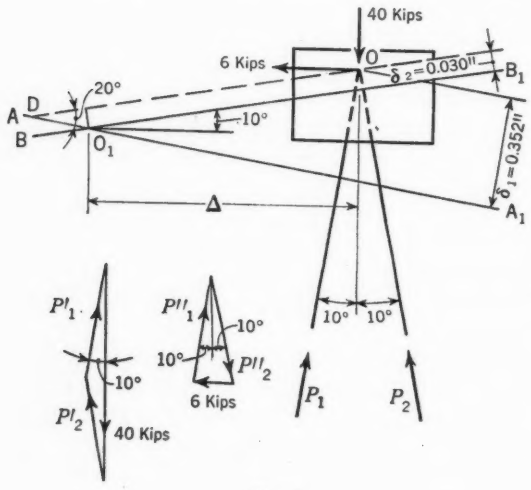


FIG. 1

Fig. 1 represents in a simplified form a foundation supported by two piles inclined in opposite directions at 10° with the vertical. The foundation carries a vertical load of 40 kips and a horizontal load of 6 kips, applied as shown. The lateral pile resistance is disregarded. It is assumed that the allowable axial load on each pile is 40 kips, and that the axial displacement of the pile head produced by this load is $\frac{3}{8}$ in. The latter statement implies that displacement of the foundation in excess of $\frac{3}{8}$ in. is objectionable.

[2] "Lateral Pile-Loading Tests," by L. B. Feagin, *Transactions*, ASCE, Vol. 102, 1937, p. 236.

By constructing two separate force triangles, one for horizontal loads and one for vertical loads, and then by combining the forces, the two pile loads are

$$P_1 = \tfrac{1}{2} \times 40 \times \sec 10° + \tfrac{1}{2} \times 6 \times \sec 80° = 37.6 \text{ kips}$$

and

$$P_2 = \tfrac{1}{2} \times 40 \times \sec 10° - \tfrac{1}{2} \times 6 \times \sec 80° = 3.2 \text{ kips}$$

Inasmuch as both P_1 and P_2 are compression loads and are less than 40 kips, the pile loads are satisfactory.

Consider now the horizontal displacement of the foundation. Assuming that the deformation is proportional to the load, the axial displacements of the pile heads are found to be, for pile 1, $\delta_1 = \frac{3}{8} \times \frac{37.6}{40} = 0.352$ in., and, for pile 2, $\delta_2 = \frac{3}{8} \times \frac{3.2}{40} = 0.030$ in.

The deflected position O_1 of the point of intersection of the piles (Fig. 1) is found at the intersection of the line AA_1, normal to pile 1 and located at 0.352 in. below O, and the line BB_1, normal to pile 2 and situated at 0.030 in. below O.

Evidently, $OD = 0.352 \operatorname{cosec} 20°$ and $DO_1 = 0.030 \operatorname{cosec} 20°$, so that the horizontal displacement of the foundation is $\Delta = (0.352 - 0.030) \times \operatorname{cosec} 20° \times \cos 10° = 0.928$ in., which is much greater than $\frac{3}{8}$ in. and therefore is inadmissible.

The large value of this displacement is, of course, the direct result of the small batter of the piles; it would decrease with an increase in the batter. However, practical considerations of pile driving preclude too great a batter, and the situation presented herein appears typical. Thus, the fact that the axial pile loads are moderate is no guarantee that the displacement of the foundation is small and the design adequate, unless the lateral movement is small also, or unless the lateral pile loads are moderate in comparison with the allowable values, based on small displacements.

Existing Methods of Analysis

Before presenting the method proposed in this paper it is desirable to outline briefly predecessors of the method.

In the method developed by Swedish engineers and presented in the United States by C. P. Vetter,[3] M. ASCE, the lateral and rotational pile resistances R and M, acting on the ends of piles (Fig. 2), are determined by assuming the pile to act as a free beam fixed at some unknown depth h. This assumption has two weaknesses: (1) No method is suggested for determining the unknown depth h; and (2) the pile is actually loaded along its entire length by the side pressure of earth, and it does not resemble a free beam of span h. Also, no section of the pile is fixed both in position and direction, as is assumed with regard to the section at A. The effect of these erroneous assumptions is that no single value of h can result in true values of both end resistances M and R corresponding to different displacements of the pile head. No estimate of the error resulting from these incorrect assumptions seems to be possible.

[3] "Design of Pile Foundations," by C. P. Vetter, *Transactions*, ASCE, Vol. 104, 1939, p. 758.

The method of pile design[4] of Carl Culmann is based on the resolution of P_{R0}, the resultant force acting on the foundation, into three component forces (Fig. 3), acting in the directions of the piles and passing through the centers of the respective pile groups. This method possesses the following weaknesses:

a. By assumption, the foundation must be supported by three nonparallel groups of piles, with one or several parallel piles in each group. From a practical point of view the requirement of the three nonparallel groups of piles may be an unnecessary complication.

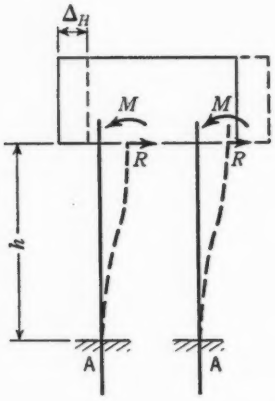


FIG. 2

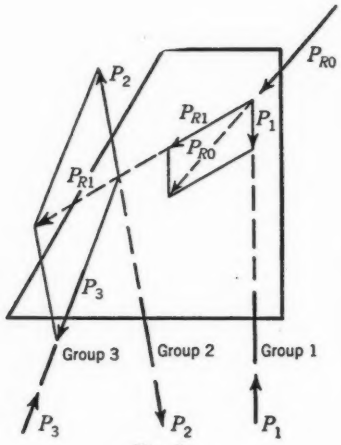


FIG. 3

b. The piles are assumed to develop axial forces only, and since no analysis of lateral deformations is made, the method is subject to the weakness previously discussed in connection with Fig. 1.

c. Each pile group is replaced in analysis by a resultant pile located at the center of gravity of the pile group. When several piles are present in each group, this assumption ignores the rotational resistances of the pile groups, developed by virtue of spreading the piles over an area, and thereby underestimates the strength of the foundation.

The method of H. M. Westergaard,[5] M. ASCE, also ignores the lateral pile resistances, and is thus subject to the same criticism.

ASSUMPTIONS OF PROPOSED METHOD

The method presented in this paper is based on the following assumptions:

1. The load carried by each pile is proportional to the displacement of the pile head. In the most general case this displacement consists of three components: The axial (in relation to the pile) displacement, δ; the transverse displacement, δ_t; and the rotational displacement, α. Although the displace-

[4] "Theoretical Soil Mechanics," by K. Terzaghi, John Wiley & Sons, Inc., New York, N. Y., 1943, p. 363.

[5] "The Resistance of a Group of Piles," by H. M. Westergaard. *Journal*, Western Soc. of Engrs., Vol. 22, 1917, p. 704.

ments thus defined are proportional to the forces producing them, they need not necessarily be considered as elastic.

2. All piles behave alike with regard to the load-deformation relation.

3. The footing, in which the pile heads are embedded, is absolutely rigid.

4. The problem is two-dimentional—that is, the piles, as well as the external forces, are arranged in planes transverse to the length of the foundation, and they are symmetrical with regard to the transverse middle plane. Under these conditions all the pile movements take place in transverse planes.

5. The footing movements are small.

It is of course fully realized that none of these assumptions is exactly true, but it is believed that in most cases they do not misrepresent the actual behavior of the piles greatly, and, at the same time, no reasonably workable analysis can be developed without their assistance. When physical conditions in the foundation manifestly disagree with any of these assumptions—for example, when the footing is not rigid—the theory, of course, fails. The same assumptions have been used by Mr. Vetter and Professor Westergaard.

The Pile Constants

The pile constants are defined as the forces with which the pile acts on the foundation when the pile head is given a unit displacement. There are three sets of these constants, corresponding to three different kinds of displacements.

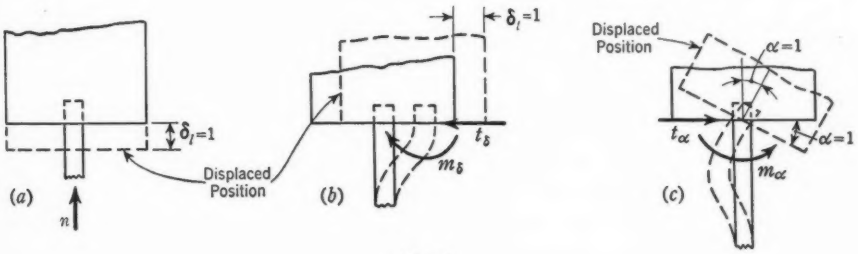


Fig. 4

Unit longitudinal displacement, $\delta_l = 1$ in. (Fig. 4(a)), brings into play a single constant n which is measured in pounds per inch and acts in the axial direction.

A unit transverse movement, $\delta_t = 1$ in. (Fig. 4(b)), produces a transverse resistance t_δ and a rotational resistance m_δ, measured in pounds per inch and pound-inches per inch, respectively.

A unit rotation, $\alpha = 1$ radian (Fig. 4(c)), is accompanied by a rotational resistance m_α and a transverse resistance t_α, measured in pound-inches per radian and pounds per radian, respectively. It may be observed that no axial resistances accompany either transverse or rotational displacements, and no moments or transverse forces are brought into play by the axial displacement.

It should be emphasized that the pile constants as defined herein represent the effect of the pile on the footing and not vice versa. Therefore, the directions of these constants must be carefully noted. As may be expected, the

three primary constants (n, t_δ, and m_α), are directed opposite to the pile head displacements creating them, whereas the two secondary constants (m_δ and t_α), which are only incidental to the displacements produced, stand in such a relation to their primary partners (that is, to t_δ and m_α, respectively) that, if the t constant is directed to the right, the m constant is directed counterclockwise, and vice versa.

By the Betti theorem,[6] $t_\alpha = m_\delta$, so that in the most general case—that is, when the pile head is deeply embedded in the footing—there are only four constants characterizing the load-deformation relation of a pile—n, t_δ, $m_\delta = t_\alpha$, and m_α.

When the footing barely engages the piles, their heads may be considered as hinged instead of fixed, and $m_\delta = 0$. Likewise, the two α constants need not be considered in this case, and only n and t_δ remain.

The Foundation Constants

Fig. 5 represents the foundation of a structure supported by piles and referred to rectangular axes. For convenience, the x-axis is made coincident with the footing base; the positive directions of the axes are taken to the right and downward, and the origin is chosen arbitrarily. The foundation constants are defined as the resultant forces with which all piles act together on the footing, when the footing is given a unit translational displacement in the positive direction of one of the axes or a unit rotation about the origin in the clockwise direction.

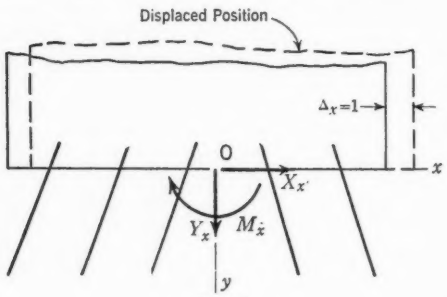


Fig. 5

Each unit displacement in general brings into play a resisting force, which may be represented by its two components acting along the coordinate axes and the moment about the origin. Thus, unit displacement along the x-axis gives rise to the constants X_x, Y_x, and M_x, in Fig. 5. A unit displacement along the y-axis creates the constants X_y, Y_y, and M_y, whereas a unit rotation about the origin produces the constants X_α, Y_α, and M_α. By the Betti law,[6] three pairs of these constants are equal, thus leaving only six independent values: X_x, $Y_x = X_y$, $M_x = X_\alpha$, Y_y, $M_y = Y_\alpha$, and M_α. The positive signs of these functions correspond to the positive directions of the axes and to the clockwise direction for the moments.

These foundation constants can be evaluated in terms of the pile constants and the geometry of the foundation. In order to avoid errors it is necessary to adhere strictly to the sign convention previously explained, keeping in mind that the foundation constants represent the action of the piles on the foundation, and not the opposite effect.

[6] "Theory of Statically Indeterminate Structures," by W. M. Fife and J. B. Wilbur, McGraw-Hill Book Co., Inc., New York, N. Y., 1937, p. 39.

Analysis of Pile Foundation

In order to determine the pile loads produced by the external forces acting on the structure it is necessary to find the three component displacements of the foundation: The displacement Δ_x along the x-axis; the displacement Δ_y along the y-axis; and the angle of rotation, α. These displacements are found by setting up three simultaneous equations expressing the equilibrium of the foundation under the action of the external forces and the pile forces induced by these displacements.

Let the components of the external forces, acting in the positive directions of the axes, and their moment about the origin, be X, Y, and M. The forces induced by the displacement Δ_x are then $X_x \Delta_x$ and $Y_x \Delta_x$ along the x-axis and the y-axis, respectively; and $M_x \Delta_x$, a moment about the origin.

The forces produced by the displacements Δ_y and α are expressed in a similar manner. The equations of equilibrium of the footing are then

$$X_x \Delta_x + X_y \Delta_y + X_\alpha \alpha + X = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1a)$$

$$X_y \Delta_x + Y_y \Delta_y + Y_\alpha \alpha + Y = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1b)$$

and

$$X_\alpha \Delta_x + Y_\alpha \Delta_y + M_\alpha \alpha + M = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1c)$$

from which the component displacements of the footing—Δ_x, Δ_y, and α—are determined. Once these component displacements are known, the displacements of the individual pile heads are found by geometry, and from them the axial, lateral, and rotational pile loads are easily computed.

Principal Axes

In a possible modification of the procedure the coordinate axes, together with their origin, may be moved to a position outside the footing base, so located that a displacement of the footing along one of the axes would produce a force on the footing acting exactly along that axis, and the rotation about the origin would produce a moment without any force. The similarity of this approach to the analysis of fixed-ended arches by means of the elastic center may be easily recognized. In this case, the three displacements of the foundation may be found directly without solving the simultaneous equations. Although apparently simple in principle, this approach has been found quite laborious in practice and for this reason is not pursued herein.

Expressions for the Foundation Constants

Fig. 6 represents a foundation supported by several nonparallel groups of piles. Group 1 includes N_1 piles making an angle ϕ_1 with the foundation base; group 2 includes N_2 piles at an angle ϕ_2, and so on—the total number of pile groups being g. The angle ϕ is measured from the positive direction of the x-axis clockwise to the given pile.

The footing is displaced a unit distance parallel to itself in the positive direction of the x-axis. In view of the rigid embedment, the piles retain their original inclinations at the ends and the head of a typical pile, A, has moved to

A_1, a distance $\cos \phi_1$ along the pile and $\sin \phi_1$ across the pile away from point A. These displacements induce pile A to act on the foundation with the following forces, applied at the point A_1 or A, and shown in their proper directions in Fig. 6: A force, $n \cos \phi_1$, along the pile; a force, $t_\delta \sin \phi_1$, across the pile; and a moment, $m_\delta \sin \phi_1$. Similar forces are evidently applied at the heads of the other piles, and the only difference in their algebraic expressions is in the subscript of the angle ϕ.

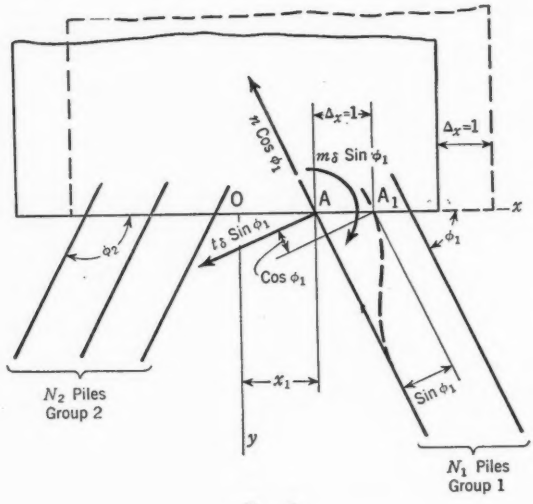


FIG. 6

The sums of the components of all the induced pile forces in the directions of the axes represent the foundation constants X_x and Y_x. The part of X_x contributed by the two resistances of pile A is, with its proper sign, $-(n \cos^2 \phi_1 + t_\delta \sin^2 \phi_1)$. For all N_1 piles of the same group this contribution is N_1 times greater. Summing up all g groups, the corresponding foundation constant is

$$X_x = -\sum_g [N (n \cos^2 \phi + t_\delta \sin^2 \phi)] \ldots\ldots\ldots\ldots\ldots\ldots (2a)$$

in which the symbol $\sum_g$ covers g terms of the form expressed by the brackets.

The contribution of pile A to Y_x is $t_\delta \sin \phi_1 \cos \phi_1 - n \cos \phi_1 \sin \phi_1$, or $-\frac{1}{2}(n - t_\delta) \sin 2\phi_1$. Thus,

$$Y_x = -\tfrac{1}{2}(n - t_\delta) \sum_g (N \sin 2\phi) \ldots\ldots\ldots\ldots\ldots\ldots (2b)$$

The moment produced by the resistances of pile A about the origin is $Y_x x_1 + m_\delta \sin \phi_1$, or $-\frac{1}{2}(n - t_\delta) \sin 2\phi_1 (x_1) + m_\delta \sin \phi_1$. The moment of all the piles of group 1 about the origin is $-\frac{1}{2}(n - t_\delta) \sin 2\phi_1 (x_1 + x_2 + \cdots + x_{N1}) + N_1 m_\delta \sin \phi_1$, or $-\frac{1}{2}(n - t_\delta) N_1 \bar{x}_1 \sin 2\phi_1 + N_1 m_\delta \sin \phi_1$, in which $\bar{x}_1$ is the abscissa of the center of gravity of all the pile heads in group 1. Summing up these moments for all g groups,

$$M_x = -\tfrac{1}{2}(n - t_\delta) \sum_g (N \bar{x} \sin 2\phi) + m_\delta \sum_g (N \sin \phi) \ldots\ldots\ldots (2c)$$

in which $\bar{x}$ is the coordinate of the center of the group of N piles having the angle ϕ.

Fig. 7 presents the diagram needed for determining the Y constants of the foundation. The footing is moved a unit distance parallel to itself in the positive direction of the y-axis, so that the head of a typical pile of group 1 is moved from A to A_1, a distance $\sin \phi_1$ along the pile and $\cos \phi_1$ across the pile. The forces induced in the pile by these displacements are shown in their proper directions in Fig. 7.

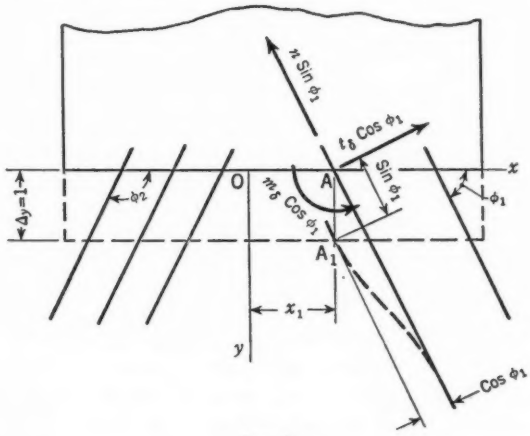


FIG. 7

Summation of the Y components of all the pile forces, and of their moments about the origin, gives

$$Y_y = -\sum_g [N\,(n \sin^2 \phi + t_\delta \cos^2 \phi)] \ldots (3a)$$

and

$$M_y = -\sum_g (n \sin^2 \phi + t_\delta \cos^2 \phi)\, N \bar{x} - m_\delta \sum_g (N \cos \phi) \ldots (3b)$$

In Fig. 8 the footing is rotated through a unit angle in the clockwise direction about the origin. The head of a typical pile A, a distance x_1 from the origin, moves vertically downward through a distance x_1. (The correct statement would be that a rotation $d\alpha$ results in the displacement $x_1\, d\alpha$.) This displacement is resolved into the components $x_1 \sin \phi_1$ along the pile and $x_1 \cos \phi_1$ across the pile. Because of these displacements the pile exerts forces on the foundation—$n\, x_1 \sin \phi_1$ along the pile, and $t_\delta\, x_1 \cos \phi_1$ across the pile—in addition to the moment, $m_\delta\, x_1 \cos \phi_1$, about the origin. The pile head also undergoes a unit rotation, which brings into play the transverse force t_α and the moment m_α in the directions shown in Fig. 8. Taking the sum of the contributions of all piles to the moment about the origin, and keeping in mind that $m_\delta = t_\alpha$, the remaining foundation constant is found to be

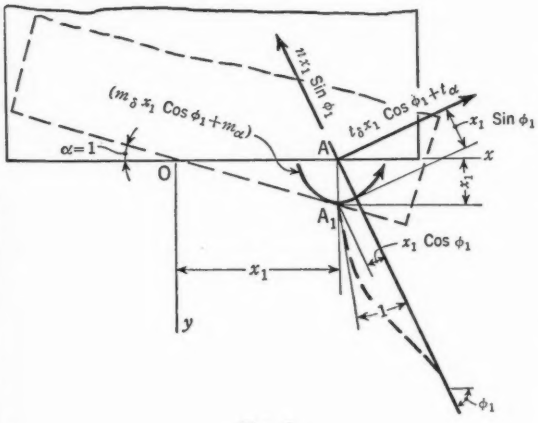


FIG. 8

$$M_\alpha = -\sum_g [(n \sin^2 \phi + t_\delta \cos^2 \phi) \sum_N (x^2)] - 2\, m_\delta \sum_g (N \bar{x} \cos \phi) - m_\alpha \sum_g (N) \ldots (4)$$

The foundation constants determined by Eqs. 2, 3, and 4, with their proper signs, represent forces and moments exerted by the piles on the footing and not by the footing on the piles. Positive signs of these functions indicate forces acting in the positive directions of the coordinate axes or, in the case of moments, in the clockwise direction. The x coordinates and the trigonometric functions in their expressions must be taken with proper algebraic signs.

Eqs. 2, 3, and 4 presuppose full embedment of the piles in the footing, which makes them behave as if they had fixed ends. When the embedment is small, a pin-ended condition ensues, and the corresponding formulas for the pile constants may be found by substituting $m_\delta = m_\alpha = 0$ in Eqs. 2, 3, and 4.

Pile Forces

Substitution of the values of the foundation constants and of the external forces X, Y, and M acting on the structure in Eqs. 1 results in determination of the foundation displacements Δ_x along the x-axis, Δ_y along the y-axis, and the angle of rotation α. Positive values of Δ_x, Δ_y, and α correspond to positive directions of the axes and clockwise rotation.

Prior to determining the pile forces it is necessary to obtain the expressions for the pile head displacements. This is done in relation to an arbitrary pile with a coordinate x_1 (see Figs. 6, 7, and 8) as follows:

The longitudinal displacement downward,

$$\delta_l = \Delta_x \cos\phi_1 + \Delta_y \sin\phi_1 + \alpha\, x_1 \sin\phi_1 \quad \ldots\ldots\ldots\ldots (5a)$$

the transverse displacement to the right,

$$\delta_t = \Delta_x \sin\phi_1 - \Delta_y \cos\phi_1 - \alpha\, x_1 \cos\phi_1 \quad \ldots\ldots\ldots\ldots (5b)$$

and the rotation clockwise, α.

The expressions for the pile forces are as follows:

The longitudinal force (compression),

$$P = n\, \delta_l \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6a)$$

the transverse force, acting on the foundation to the right,

$$Q = -\, t_\delta\, \delta_t + m_\delta\, \alpha \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6b)$$

and the moment, acting on the foundation clockwise,

$$S = m_\delta\, \delta_t - m_\alpha\, \alpha \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

Solution by Ratios

Although the expressions for the pile loads (Eqs. 6 and 7) are functions of the pile constants, it is important to realize that it is not the constants themselves but their ratios that determine the pile loads, as may be made clear from the following. Suppose that all four pile constants decrease in the same ratio q. Then, as may be seen from Eqs. 2, 3, and 4, all the foundation constants are reduced q times. The result is that the foundation displacement, determined

from Eqs. 1, and the pile head displacements δ_l, δ_t, and α (Eqs. 5), all become q times greater. Turning now to the pile forces, it may be seen that Eqs. 6 and 7 are made up of the products of the pile constants and the pile displacements, and because the former functions decrease q times and the latter increase q times, the pile forces remain unchanged.

Thus, in the general case of fixed-ended piles, their loads are fully determined by the three ratios,

$$r_1 = \frac{t_\delta}{n} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8a)$$

$$r_2 = \frac{m_\delta}{n} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8b)$$

and

$$r_3 = \frac{m_\alpha}{n} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8c)$$

When the piles behave as having pinned ends, only one ratio, r_1, remains, r_2 and r_3 both becoming zero.

It is preferable to approach the problem by the method of ratios rather than that of pile constants, not only because of the smaller number of coefficients involved, but also because r_1, the most important of the ratios, is more easily ascertainable than the pile coefficients, as made clear later. For the purpose of explaining the ratio approach new quantities, referred to as the reduced foundation constants, are introduced. Their expressions are as follows (see Fig. 9):

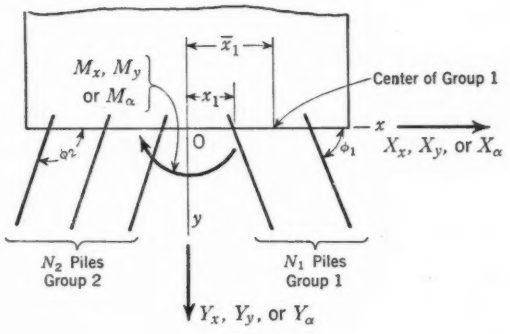


FIG. 9

$$X'_x = \frac{X_x}{n} = -\sum_q [N\,(\cos^2\phi + r_1 \sin^2\phi)] \quad \ldots\ldots\ldots\ldots (9a)$$

$$Y'_x = \frac{X_y}{n} = \frac{Y_x}{n} = -\tfrac{1}{2}\,(1 - r_1) \sum_q (N \sin 2\phi) \quad \ldots\ldots\ldots\ldots (9b)$$

$$M'_x = \frac{M_x}{n} = \frac{X_\alpha}{n} = -\tfrac{1}{2}\,(1 - r_1) \sum_q (N\,\bar{x} \sin 2\phi) + r_2 \sum_q (N \sin\phi) \ldots (9c)$$

$$Y'_y = \frac{Y_y}{n} = -\sum_q [N\,(\sin^2\phi + r_1 \cos^2\phi)] \quad \ldots\ldots\ldots\ldots (9d)$$

$$M'_y = \frac{M_y}{n} = \frac{Y_\alpha}{n} = -\sum_q [(\sin^2\phi + r_1 \cos^2\phi)\, N\,\bar{x}] - r_2 \sum_q (N \cos\phi) \ldots (9e)$$

and

$$M'_\alpha = \frac{M_\alpha}{n} = -\sum_q [(\sin^2\phi + r_1 \cos^2\phi) \sum_N (x^2)] - 2\,r_2 \sum_q (N\,\bar{x} \cos\phi) - r_3 \sum_q (N) \ldots (9f)$$

Although the foundation constants proper are unknown, the reduced constants can be easily determined, because they are expressed in terms of the pile ratios and not in terms of the pile constants. By rearranging to the form:

$$\frac{X_x}{n}(n\,\Delta_x) + \frac{Y_x}{n}(n\,\Delta_y) + \frac{M_x}{n}(n\,\alpha) + X = 0 \quad \ldots\ldots(10)$$

and substituting

$$\Delta'_x = n\,\Delta_x \quad \ldots\ldots(11a)$$

$$\Delta'_y = n\,\Delta_y \quad \ldots\ldots(11b)$$

and

$$\alpha' = n\,\alpha \quad \ldots\ldots(11c)$$

for the expressions in the brackets, named the reduced foundation movements, Eqs. 1 become

$$X'_x\,\Delta'_x + Y'_x\,\Delta'_y + M'_x\,\alpha' + X = 0 \quad \ldots\ldots(12a)$$

$$Y'_x\,\Delta'_x + Y'_y\,\Delta'_y + M'_y\,\alpha' + Y = 0 \quad \ldots\ldots(12b)$$

and

$$M'_x\,\Delta'_x + M'_y\,\Delta'_y + M'_\alpha\,\alpha' + M = 0 \quad \ldots\ldots(12c)$$

—from which Δ'_x, Δ'_y, and α' can be determined. The actual foundation movements and the pile head displacements cannot be found by this approach, but the pile loads are easily determined as follows:

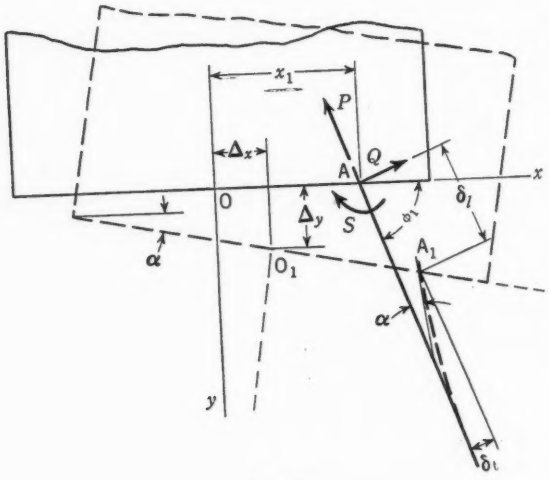


FIG. 10.

From Eqs. 5, the reduced pile head displacements are

$$\delta'_l = n\,\delta_l = \Delta'_x \cos\phi_1 + \Delta'_y \sin\phi_1 + \alpha'\,x_1 \sin\phi_1 \quad \ldots(13a)$$

$$\delta'_t = n\,\delta'_t = \Delta'_x \sin\phi_1 - \Delta'_y \cos\phi_1 - \alpha'\,x_1 \cos\phi_1 \quad \ldots(13b)$$

and α'.

From Eqs. 6 and 7 the pile loads (see Fig. 10) are given by

$$P = \delta'_l \quad \ldots(14a)$$

and

$$Q = -r_1\,\delta'_t + r_2\,\alpha' \quad \ldots(14b)$$

$$S = r_2\,\delta'_t - r_3\,\alpha' \quad \ldots(15)$$

APPROXIMATE THEORY

Determination of the numerical values of the pile ratios, as well as of the pile constants, is at best a rather uncertain procedure. Fortunately, however, the difficulty arising because of this uncertainty can be overcome by means of a moderate approximation, which at the same time adds to the practicability

of the method. Comparative calculations have shown that it is possible, without serious error, to ignore completely the ratios r_2 and r_3, and to express the remaining ratio, r_1, in terms of the data possessed by the designer.

It is interesting to note that the ratios r_2 and r_3 (or, which is just the same, the constants m_δ and m_α) are disregarded in the conventional method of analysis of vertical piles (Fig. 11). According to this method, the axial load on the extreme pile (at a distance x_1 from the origin to the pile center) is

$$P = \frac{Y}{N} + \frac{M\,x_1}{\sum x^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (16)$$

which clearly omits the effects of m_δ and m_α. Actually, the influence of m_δ is manifested in the end moments on the piles, due to the horizontal displacement of the foundation produced by the horizontal force X, thus augmenting the external moment M in its effect on the values of the longitudinal pile loads. At the same time the rotation of the footing caused by inequality of the longitudinal pile loads results in formation of the end moments, which are dependent on m_α, and somewhat reduce the vertical pile loads. Thus, the usual design practice, when applied to foundations supported by the vertical piles alone, definitely ignores the effect of r_2 and r_3 without explicitly stating so.

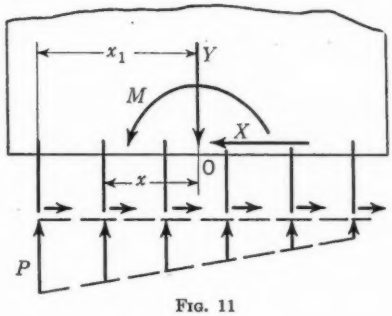


FIG. 11

Analysis of pile foundations by means of a single ratio r_1 involves omitting from the equations concerned terms containing the ratios r_2 and r_3. The moment on the end of the pile (Eq. 15) is, of course, eliminated. The complete set of equations required for the pile load analysis by the approximate theory includes, in addition to Eqs. 9*a*, 9*b*, 9*d*, and 11,

$$M'_x = X'_\alpha = -\tfrac{1}{2}(1 - r_1) \sum_g (N\,\bar{x} \sin 2\phi) \quad \ldots\ldots\ldots\ldots (17)$$

$$M'_y = Y'_\alpha = -\sum_g [(\sin^2\phi + r_1 \cos^2\phi)\, N\,\bar{x}] \quad \ldots\ldots\ldots\ldots (18)$$

$$M'_\alpha = -\sum_g [(\sin^2\phi + r_1 \cos^2\phi) \sum_N (x^2)] \quad \ldots\ldots\ldots\ldots (19)$$

and

$$Q = -r_1\,\delta'_t \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (20)$$

Eqs. 17, 18, 19, and 20, as well as Eqs. 9*a*, 9*b*, 9*d*, 13, and 14*a*, involve no approximation when the piles behave as hinged at the top ends, but the values of r_1 for the pin-ended condition and the fixed-ended condition are quite different.

If the allowable loads on the pile in the axial and transverse directions are P_a and Q_a, respectively, and the pile head displacements caused by these loads are $(\delta_a)_l$ and $(\delta_a)_t$,

$$n = \frac{P_a}{(\delta_a)_l} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (21a)$$

and

$$t_\delta = \frac{Q_a}{(\delta_a)_t} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (21b)$$

Substituting Eqs. 21 in Eq. 8*a* and assuming that for properly established allowable loads $(\delta_a)_l = (\delta_a)_t$ results in

$$r_1 = \frac{Q_a}{P_a} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (22)$$

Inasmuch as the lateral pile resistances, as well as the axial resistances, are constantly relied upon in the usual design, the values of both P_a and Q_a must be known to the designer, whether by code, by test, or by guessing. Therefore, the ratio r_1 must also be known to him. This ratio can be wrong only if unreasonable or inconsistent values have been chosen for Q_a and P_a, in which case the resultant error in all fairness cannot be charged to the method, just as the theory of reinforced concrete cannot properly be charged with the discrepancy arising in the case of a reinforced concrete structure when the design is based on the assumption of 3,000-lb concrete and the actual strength of concrete in the structure is only 1,500 lb per sq in.

The comparison with reinforced concrete is instructive also from another viewpoint. It is well known that the relation between stress and strain in concrete is neither strictly linear, even at the beginning of the stress-strain curve, nor fully elastic. Furthermore, the so-called secant modulus, even in concrete of the same strength characteristics, is a quantity quite variable for different specimens and much different from the generally accepted value of $E = 1{,}000\,f'_c$, yet the conventional design of reinforced concrete is seldom questioned with regard to the bending stresses in concrete and steel. It seems reasonable to adopt a similar attitude in the design of pile foundations. Then r_1 may be looked upon as a relative measure of pile resistances, even when the dependence of the allowable pile loads on the displacements becomes obscure.

It appears that a moderate variation in the assumed value of r_1 does not alter the values of the pile loads a great deal. If for some special reasons the assumed allowable pile loads Q_a and P_a are based on different values of the displacements δ_{at} and δ_{al}, a simple adjustment can be made in Eq. 22.

The actual values of the pile loads P and Q, found by Eqs. 14*a* and 20, should not exceed the allowable values so that

$$P \leq P_a \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (23a)$$

and

$$Q \leq Q_a \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (23b)$$

Substituting in Eq. 23*b* the expression for Q from Eq. 20, and eliminating the minus sign as immaterial, it is found that

$$r_1\,\delta'_t = \frac{Q_a}{P_a}\,\delta'_t \leq Q_a \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (24a)$$

or

$$\delta'_t \leq P_a \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (24b)$$

Compliance with Eq. 24*b*, which is equivalent to Eq. 23*b*, insures that the lateral pile load and the lateral displacement of the pile head are not excessive.

When stated in the form:

$$\delta_t \leq \frac{P_a}{n} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (24c)$$

Eq. 24*b* obviously means that the lateral pile head movement must not exceed the axial movement caused by the allowable longitudinal load. Evidently, this condition must be satisfied by piles in all cases, whether they are analyzed by a single ratio, r_1, or by all three ratios.

When the lateral pile resistance is negligible, r_1 may be assumed to be zero in the expressions for the foundation constants and for the pile loads. However, the condition expressed by Eq. 24*b* still holds in this case and warns against an excessive lateral pile movement, similar to that discussed under the heading "Pile Foundations."

In most cases the piles are nearly vertical, with the result that the value of the horizontal movement of the foundation Δ_x is fairly close to the value of the lateral pile movement δ_t; for this reason the condition:

$$\Delta'_x \leq P_a \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (25)$$

may be used as approximately equivalent to Eq. 24*b*. Thus, Eqs. 23*a* and 25, or Eqs. 23*a* and 24*b*, may be used in all cases as the criteria of adequacy of the design, in lieu of Eqs. 23.

Approximate Formulas for the Pile Constants

Determination of the pile constants n, t_δ, m_δ, and m_α or of the three ratios r_1, r_2, and r_3 need not be considered as a necessary part of the present investigation. However, in order to check the agreement of the approximate and the more exact analyses of the pile foundation, it has been found desirable to make a rough computation of these constants. The method used is admittedly crude, being based on a number of rather arbitrary assumptions. This circumstance, however, should not detract from the usefulness of the numerical investigation founded on it, because its main purpose lies not in determining the true values of the coefficients, but rather in establishing their order for different foundation conditions, with the view of demonstrating the closeness of the approximate and the more exact pile analyses under virtually all possible combinations of piles and soils. The range of the primary physical coefficients determining the values of the pile constants will be chosen so wide that the true values of the latter are not likely to go beyond the limits assumed in the investigation.

Values of n

It follows from Eq. 21*a* that small values of n occur with lightly loaded piles which settle a great deal, whereas large values of n occur when the piles are loaded heavily but their settlements are small. Allowance must be made for both the deformation of the soil under the foot of the pile and the elastic shortening of the pile. A $\frac{1}{2}$-in. axial displacement of the head under the allowable load will be considered a large settlement; the small settlement will be

taken equal to the elastic shortening of the pile alone. In the latter case a distinction will be made between a very weak soil (so soft that it offers no axial support to the pile), in which the pile is assumed to be driven to a firm stratum below so that all its axial support comes from the foot reaction, and other soils, in which the pile is assumed to be supported by skin friction alone. The pile shortenings in these two types of soils are found by the formulas:

For very weak soil,

$$(\delta_a)_l = \frac{P_a L}{A E} \quad \ldots\ldots (26a)$$

and for other soils,

$$(\delta_a)_l = \frac{P_a L}{2 A E} \quad \ldots\ldots (26b)$$

Two rather extreme classes of piles are used in the following analyses—a 30-ft timber pile, 9 in. in average diameter, and a 100-ft reinforced concrete pile, 20 in. square. The data pertaining to these piles, together with the computed values of n, are given in Table 1.

TABLE 1.—DATA AND COMPUTED VALUES OF n FOR SPECIMEN PILES

Pile	Section[a] (in.)	Area[a] (sq in.)	Modulus of elasticity, E (kips per sq in.)	Load, P_a (kips)	$(\delta_a)_l$ (IN.)			n (KIPS PER IN.)		
					Maximum	MINIMUM		Minimum	MAXIMUM	
						Very weak soil	Other soil		Very weak soil	Other soil
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
30-ft timber	9[b]	63.5	1,500	30	0.5	0.1134	0.0567	60	265	530
100-ft reinforced concrete	20[c]	400	3,000	140	0.5	0.14	0.07	280	1,000	2,000

[a] Average. [b] Round. [c] Square.

CONSTANTS t_δ, m_δ, AND m_α

Approximate values of the constants t_δ, m_δ, and m_α can be determined by considering the pile as a beam on an elastic foundation of infinite length, loaded at the free end.[7] The end conditions necessary for determining the constants

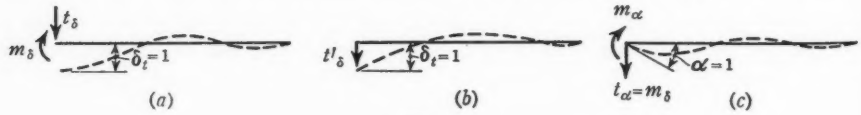


FIG. 12

t_δ and m_δ are shown in Figs. 12(*a*) and 12(*b*), in which the transverse force is t_δ and the end moment m_δ, whereas the deflection at the end is unity, and the angle change is zero. For determination of m_α, the end conditions (Fig. 12(*c*)) are: A transverse force $t_\alpha = m_\delta$, a moment m_α, zero end deflection, and unity

[7] "Strength of Materials," by S. Timoshenko, D. Van Nostrand Co., Inc., New York, N. Y., Vol. 2, 1941.

angle change. Translating the notation of S. Timoshenko into that of Fig. 12,

$$t_\delta = \frac{k}{\beta} \quad \ldots\ldots (27a)$$

$$m_\delta = \frac{k}{2\beta^2} = \frac{t_\delta}{2\beta} \quad \ldots\ldots (27b)$$

and

$$m_\alpha = \frac{k}{2\beta^3} = \frac{m_\delta}{\beta} \quad \ldots\ldots (27c)$$

in which k is the elastic constant of the soil, defined as the ratio of the load per unit length of pile and the corresponding deflection of the pile at the same point, that is,

$$k = \frac{p}{\delta_t} \quad \ldots\ldots (28)$$

Also, the coefficient

$$\beta = \sqrt[4]{\frac{k}{4EI}} \quad \ldots\ldots (29)$$

in which I and E are, respectively, the moment of inertia of the cross section and the modulus of elasticity of the pile. When the pile behaves as having pinned ends (Fig. 12(*b*)),

$$t'_\delta = \frac{k}{2\beta} = \tfrac{1}{2} t_\delta \quad \ldots\ldots (30)$$

This method involves the approximations and assumptions that: (*a*) The intensity of the earth reaction at each point of the pile is proportional to the deflection of the pile at the same point; (*b*) the cross section of the pile is constant throughout its full length; and (*c*) the pile is infinitely long. Under the last condition the greatest error occurs when the pile is short and rigid and the soil is weak. Correction for the finite pile length is possible but laborious, and calculations indicate that this error is moderate even in the extreme cases.

Estimation of the Soil Pressure Constant k

The value of the soil pressure constant depends on the stiffness of the soil, the size of the pile, and its cross-sectional shape. Three widely different classes of soil are considered: (1) Very weak, compressing 1 in. under a pressure of 100 lb per sq ft; (2) weak, compressing 1 in. under a pressure of 1 kip per sq ft; and (3) medium, compressing 1 in. under a pressure of 10 kips per sq ft. Generally speaking, soil firmer than medium is not likely to require the presence of piles, and therefore is not considered in connection with the pile constants.

A round pile, by virtue of its shape, is believed to produce a sort of wedge action on the soil. This condition is allowed for by assuming that for a round pile the deformation is double that of a square pile of the same diameter.

The computation of k may be illustrated by the following examples, both of which assume very weak soil: 20-in. square pile, $k = \frac{100 \times 20}{12 \times 12} = 13.88$ lb per sq in.; and 9-in. diameter round pile, $k = \frac{100 \times 9}{2 \times 12 \times 12} = 3.12$ lb per sq in.

The values of the pile constants t_δ, m_δ, and m_α, together with values of k and β, corresponding to different piles and soils are given in Table 2.

It is interesting to present at this point the available result of an independent determination of the pile constants. In a rather elaborate theoretical in-

TABLE 2.—VALUES OF t_δ, m_δ, m_α, k, β, n, r_1, r_2, AND FOR

Pile	Section[a] (in.)	I (in.^{4})[a]	E (kips per sq in.)	Type of soil	k (lb per sq in.)	1,000 β (in.$^{-1}$)	t_δ (kips per in.)	m_δ (in-kip per in.)	m_α (in-kip per radian
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
30-ft timber	9-in. round	322	1,500	Very weak	3.12	6.34	0.493	38	6,14
				Weak	31.2	11.28	2.77	123	10.90
				Medium	312	20.03	15.59	389	19,40
100-ft reinforced concrete	20-in. square	13,333	3,000	Very weak	13.88	3.05	4.55	745	244,00
				Weak	138.8	5.43	25.6	2,360	435,00
				Medium	1,388	9.65	143.7	7,450	772,00
									Averag

vestigation, based on the data of lateral pile loading tests performed by L. B. Feagin, M. ASCE, A. E. Cummings,[8] M. ASCE, arrives at the following values (in this paper called the pile constants) for a round timber pile of 12-in. average diameter, with $E = 1.878$ kips per sq in., in the soil described as medium sand—$t_\delta = 32$ kips per in. and $m_\delta = 1{,}200$ in-kip per in. Reduction of the constants of the 9-in.-diameter timber piles in Table 2 to comparable sizes and stiffnesses gives resultant values of $t_\delta = 27.3$ kips per in. and $m_\delta = 895$ in-kip per in. With all the uncertainties of the underlying theories and the indefiniteness of the soil designation, the agreement of these two sets of constants must be considered as close.

The values of the three r ratios are also given in Table 2, two sets of ratios being given for each class of soil and each type of pile. The maximum values correspond to the least values of n in Table 1; the minimum values, to the greatest values of n.

NUMERICAL EXAMPLE

The theory presented herein is applied to the analysis of the pile foundation under the retaining wall shown in Fig. 13. The wall is supported by transverse rows of piles at 3-ft centers along the wall, five piles per row, the two back piles being vertical and the three front ones inclined at a batter of 3 on 1. The earth pressure is found by Rankine's theory, using the following data: The unit weight of concrete is 150 lb per cu ft; w, the unit weight of soil, is 100 lb per cu ft; and θ, the angle of friction of the soil, is 30°. The earth pressure at the back is represented by the active pressure

$$E_a = \tfrac{1}{2}\, w\, h^2 \frac{1 - \sin\theta}{1 + \sin\theta} \qquad (31a)$$

[8] Discussion by A. E. Cummings of "Lateral Pile-Loading Tests," by L. B. Feagin, *Transactions*, ASCE, Vol. 102, 1937, p. 255.

At the front it is represented by one half of the passive pressure or

$$\tfrac{1}{2} E_p = \tfrac{1}{2} (\tfrac{1}{2})\, w\, h^2{}_1 \frac{1 + \sin\theta}{1 - \sin\theta} \qquad (31b)$$

Both these pressures are horizontal and, to avoid complicating the problem by

FOR SPECIMEN PILES IN VARIOUS TYPES OF SOILS

m_β (in-kip per in.)	m_α (in-kip per radian)	n (KIPS PER IN.)		r_1		r_2 (IN.)		r_3 (IN.²)	
		Maximum	Minimum	Minimum	Maximum	Minimum	Maximum	Minimum	Maximum
(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)
38	6,140	265	60	0.00186	0.00821	0.1468	0.648	23.16	102.3
123	10,900	530	60	0.00523	0.0462	0.232	2.05	20.57	181.9
389	19,400	530	60	0.0294	0.260	0.734	6.48	36.6	323.3
745	244,000	1,000	280	0.00455	0.01625	0.745	2.66	244	872
2,360	435,000	2,000	280	0.0128	0.0913	1.180	8.43	217.5	1,555
7,450	772,000	2,000	280	0.0719	0.513	3.725	26.6	386	2,760

Average.

further uncertainties of soil mechanics, are assumed to have the same values in all cases of pile-soil combination, irrespective of the displacements of the foundation and structure. The origin of the coordinates is placed at the central pile and the external forces acting on the 3-ft section of the wall, referred to the origin, are shown in Fig. 13.

The problem is solved repeatedly using all twelve sets of the pile ratios and considering first that all the ratios r_1, r_2, and r_3 are effective (cases 2 to 13, Table 3). Then the ratio r_1 alone is considered to be effective (cases 2*a*, to 13*a*). In addition, the problem is solved by disregarding all pile resistances other than axial (case 1).

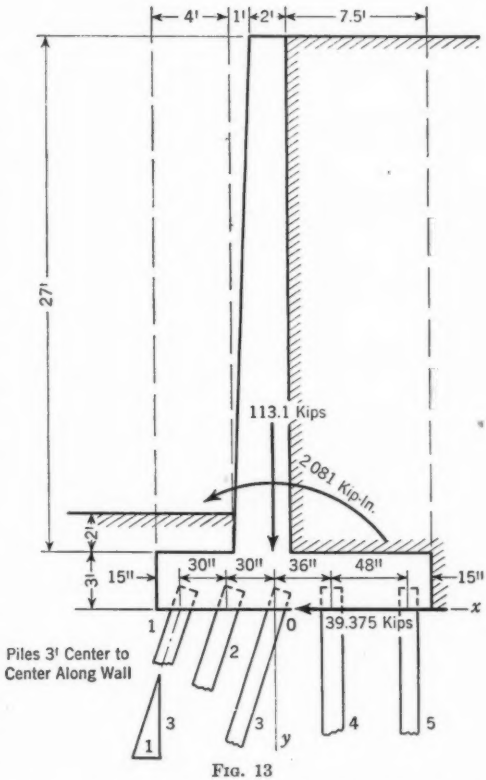


FIG. 13

Employment of large and closely spaced reinforced concrete piles in a structure of this size may be criticized as incongruous. Nevertheless, although such piles would admittedly be too large for the conditions indicated, the analysis based on the ratios peculiar to them may be looked upon as the one

presenting the picture of mechanical behavior of these piles in larger and more heavily loaded structures. As far as the structure in question is concerned, the piles endowed with the ratios corresponding to the large reinforced concrete piles will be considered as having the same moderate allowable load as the timber piles. This load in all cases is taken as $P_a = 40$ kips.

TABLE 3.—PILE LOADS FOR WALL OF FIG. 1

Pile	Type of soil	Case No.	Pile Ratios			Reduced Foundation Movements			Pile No. 1			Pile No. 2
			r_1	r_2 (in.)	r_3 (in.²)	Δ^1_x (kips)	Δ^1_y (kips)	α^1 (kips per in.)	P (kips)	Q (kips)	S (in-kip)	P (kips)
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)
		1	0	0	0	−311.1	−40.8	+0.638	+23.5	....	....	+41.0
Timber, 9 in. round 30 ft long	Very weak	2	0.00186	0.1468	23.16	−212.6	−19.32	+0.3384	+29.7	+0.45	−39	+39.4
		2a	0.00186	....	....	−233.0	−23.84	+0.413	+27.5	+0.44	....	+39.3
		3	0.00821	0.648	102.3	−113.6	+2.136	+0.0405	+35.6	+0.91	−74	+36.7
		3a	0.00821	....	....	−129.9	−1.603	+0.1176	+32.9	+1.03	....	+36.3
	Weak	4	0.00523	0.232	20.57	−149.6	−5.74	+0.1590	+32.8	+0.80	−37	+37.4
		4a	00.0523	....	....	−162.3	−8.58	+0.2101	+31.2	+0.84	....	+37.2
		5	0.0462	2.05	181.9	−42.4	+17.44	−0.1525	+38.6	+1.16	−37	+34.3
		5a	0.0462	....	....	−39.0	+18.05	−0.1429	+37.6	+1.32	....	+33.8
	Medium	6	0.0294	0.734	36.6	−54.8	+14.75	−0.1108	+37.6	+1.25	−29	+34.8
		6a	0.0294	....	....	−54.9	+14.64	−0.0973	+36.8	+1.34	....	+34.6
		7	0.260	6.48	323.3	−17.50	+22.55	−0.2043	+38.5	+0.13	+30	+32.2
		7a	0.260	....	....	−12.65	+23.57	−0.2158	+38.6	+0.12	....	+32.3
Reinforced concrete, 20 in. square 100 ft long	Very weak	8	0.00455	0.745	244	−135.5	−2.448	+0.0854	+35.7	+0.65	−118	+38.1
		8a	0.00455	....	....	−173.2	−10.92	+0.2417	+30.7	+0.78	....	+37.5
		9	0.01625	2.66	872	−78.0	+10.02	−0.0805	+38.7	+0.91	−114	+36.4
		9a	0.01625	....	....	−83.3	+8.52	−0.0162	+35.4	+1.24	....	+34.9
	Weak	10	0.0128	1.18	217.5	−87.4	+7.89	−0.0407	+37.4	+0.97	−85	+36.3
		10a	0.0128	....	....	−97.8	+5.39	+0.0254	+34.6	+1.17	....	+35.3
		11	0.0913	8.43	1,555	−36.58	+18.57	−0.1561	+38.0	+1.04	+24	+33.6
		11a	0.0913	....	....	−23.98	+21.24	−0.1852	+38.3	+1.14	....	+33.1
	Medium	12	0.0719	3.725	386	−34.65	+19.08	−0.1720	+38.8	+1.05	−21	+33.9
		12a	0.0719	....	....	−28.12	+20.39	−0.1737	+38.1	+1.22	....	+33.2
		13	0.513	26.6	2,760	−17.70	+21.92	−0.1582	+35.5	−0.68	+253	+30.9
		13a	0.513	....	....	− 9.59	+23.91	−0.2196	+38.3	−1.35	....	+32.6

The method of analysis of the pile loads is illustrated by the following example (case 2, Table 3), in which the minimum pile ratios are assumed for the timber pile in very weak ground. From Table 3, $r_1 = 0.00186$; $r_2 = 0.1468$ in.; and $r_3 = 23.16$ in. For the first group of piles: $N_1 = 3$; $\phi_1 = 108° 26'$; $\sin \phi_1 = \frac{3}{\sqrt{10}} = 0.94869$; $\cos \phi_1 = -\frac{1}{\sqrt{10}} = -0.31623$; and $\sin 2\phi_1 = -0.6$. For the second group of piles: $N_2 = 2$; $\sin \phi_2 = 1$; $\cos \phi_2 = 0$; $\sin 2\phi_2 = 0$; and $\phi_2 = 90°$.

Thus,

$$X'_x = -[3\,(0.1 + 0.00186 \times 0.9) + 2 \times 0.00186] = -0.30874$$

$$X'_y = Y'_x = -\tfrac{1}{2}\,(1 - 0.00186)\,3\,(-0.6) = 0.89833$$

$$M'_x = X'_\alpha = -0.89833\,(-30) + 0.1468\,(3 \times 0.94869 + 2) = -26.239 \text{ in.}$$

$$Y'_y = -[3\,(0.9 + 0.00186 \times 0.1) + 2 \times 1] = -2.7006 - 2 = -4.7006$$

UNDER VARIOUS COMBINATIONS OF PILE RATIOS

PILE LOADS

PILE No. 2			PILE No. 3			PILE No. 4			PILE No. 5		
P (kips)	*Q* (kips)	*S* (in-kip)	*P* (kips)	*Q* (kips)	*S* (in-kip)	*P* (kips)	*Q* (kips)	*S* (in-kip)	*P* (kips)	*Q* (kips)	*S* (in-kip)
(13)	(14)	(15)	(16)	(17)	(18)	(19)	(20)	(21)	(22)	(23)	(24)
+41.6	....	....	+59.8	....	....	−17.8	....	....	+12.7	....	....
+39.4	+0.44	−39	+49.0	+0.44	−38	−7.1	+0.45	−39	+9.1	+0.45	−39
+39.3	+0.43	....	+51.0	+0.43	....	−9.0	+0.43	....	+10.9	+0.43	....
+36.7	+0.91	−74	+37.9	+0.90	−73	+3.6	+0.96	−78	+5.5	+0.96	−78
+36.3	+1.02	....	+39.6	+1.02	....	+2.6	+1.07	....	+8.3	+1.07	....
+37.4	+0.80	−37	+41.9	+0.79	−37	0.0	+0.82	−38	+7.6	+0.82	−38
+37.2	+0.83	....	+43.2	+0.82	....	−1.0	+0.85	....	+9.1	+0.85	....
+34.3	+1.23	−40	+30.0	+1.29	−43	+12.0	+1.65	−59	+4.6	+1.65	−59
+33.5	+1.39	....	+29.5	+1.45	....	+12.9	+1.74	....	+6.1	+1.74	....
+34.5	+1.28	−30	+31.3	+1.31	−31	+10.8	+1.53	−36	+5.5	+1.53	−36
+34.0	+1.37	....	+31.2	+1.39	....	+11.1	+1.61	....	+6.5	+1.61	....
+32.7	+0.63	+17	+26.9	+1.13	+5	+15.2	+3.22	−47	+5.4	+3.22	−47
+32.5	+0.65	....	+26.4	+1.18	....	+15.8	+3.29	....	+5.4	+3.29	....
+38.1	+0.65	−118	+40.6	+0.64	−117	+0.6	+0.67	−122	+4.7	+0.67	−122
+37.5	+0.77	....	+44.4	+0.76	....	−2.2	+0.79	....	+9.4	+0.79	....
+36.4	+0.92	−116	+34.2	+0.94	−118	+7.1	+1.05	−137	+3.3	+1.05	−137
+34.9	+1.24	....	+34.4	+1.24	....	+7.9	+1.35	....	+7.2	+1.35	....
+36.3	+0.98	−86	+35.1	+0.98	−86	+6.4	+1.07	−94	+4.5	+1.07	−94
+35.3	+1.17	....	+36.0	+1.15	....	+6.3	+1.25	....	+7.5	+1.25	....
+33.6	+1.18	+12	+29.2	+1.31	−1	+12.9	+2.02	−66	+5.5	+2.02	−66
+33.1	+1.30	....	+27.8	+1.46	....	+14.6	+2.19	....	+5.7	+2.19	....
+33.9	+1.17	−27	+29.0	+1.29	−34	+12.9	+1.85	−63	+4.6	+1.85	−63
+33.2	+1.34	....	+28.2	+1.45	....	+14.1	+2.02	....	+5.8	+2.02	....
+30.9	+0.09	+213	+26.5	+0.85	+174	+16.3	+4.87	−34	+8.7	+4.87	−34
+32.0	−0.28	....	+25.8	+0.79	....	+16.1	+4.90	....	+5.5	+4.90	....

$$M'_y = Y'_\alpha = -2.7006\,(-30) - 2 \times 60 - 0.1468 \times 3\,(-0.31623) = -38.843 \text{ in.}$$

$$M'_\alpha = -[(0.9 + 0.00186 \times 0.1)\,((-30)^2 + (-60)^2) + 36^2 + 84^2] - 2 \times 0.1468 \times 3\,(-30)\,(-0.31623) - 23.16\,(3 + 2) = -12{,}527 \text{ in.}^2$$

The simultaneous equations are, therefore,

$$-0.30874\,\Delta'_x + 0.89833\,\Delta'_y - 26.239\,\alpha' - 39.375 = 0$$

$$0.89833\,\Delta'_x - 4.7006\,\Delta'_y - 38.843\,\alpha' + 113.1 = 0$$

$$-26.239\,\Delta'_x - 38.843\,\Delta'_y - 12{,}527\,\alpha' - 2{,}081 = 0$$

from which the reduced foundation displacements are found to be

$$\alpha' = +0.3384 \text{ kip per in., } \Delta'_y = -19.32 \text{ kips, and } \Delta'_x = -212.6 \text{ kips.}$$

The reduced displacements of the head of pile 1 are $\delta'_1 = -212.6\,(-0.31623) - 19.32 \times 0.94869 + 0.3384\,(-60)\,0.94869 = +29.72$ kips; $\delta'_4 = -212.6 \times 0.94869 + 19.32\,(-0.31623) - 0.3384\,(-60)\,(-0.31623) = -214.43$ kips; and $\alpha' = +0.3384$ kips per in. The loads on this pile are $P = 29.72$ kips; $Q = -0.00186\,(-214.43) + 0.1468 \times 0.3384 = 0.45$ kips; and $S = 0.1468 \times (-214.43) - 23.16 \times 0.3384 = -39.3$ in.-kip. Assuming the allowable longitudinal load on the pile to be 40 kips, the criteria of adequacy of the foundation design (Eqs. 23*a* and 25) show that the value of P is satisfactory, but that the value of Δ'_x is excessive.

All the data pertaining to the foundation movements and the pile loads, obtained by repeated solutions of the problem by different sets of the pile ratios, are presented in Table 3. The results have been verified by evaluating all the pile loads and by checking them by means of the three equations of statics.

Discussion of the Solution Results

It may be observed at the outset that the range of relative stiffnesses of piles in axial and lateral directions, implied in the values of various sets of pile ratios in Table 3, is very wide, extending all the way from zero value of lateral resistance in case 1 to a lateral resistance value more than half as great as the axial resistance in case 13 ($r_1 = 0.513$). Some interesting and unexpected results become apparent from Table 3.

a. The most striking feature is the excessive amount of horizontal displacement of the foundation, as indicated by the value of Δ'_x, which quantity greatly exceeds $P_a = 40$ kips in case of very weak and partly weak soils. Therefore, according to the lateral displacement criterion, Eq. 25, the foundation of Fig. 13 must be deemed inadequate, except in medium soil. This result may appear unexpected to many engineers.

b. Increase in the lateral stiffness of piles always leads to an increase in the lateral pile loads Q, as may well be expected. These loads are quite small, however, usually of the order of 0.5 kip to 1.5 kips per pile, so that their sum amounts to from only 2.5 kips to 7.5 kips per bent, against the external lateral load of 39.375 kips. Although the effect of the lateral pile resistances on the carrying capacity of the bent is small, its effect on the lateral rigidity of the foundation is tremendous. Thus, when the piles have no lateral resistance (case 1, Table 3) $\Delta'_x = -311.1$ kips, but when (case 6*a*) $r_1 = 0.0294$ and $r_2 = r_3 = 0$ (that is, when the lateral pile resistance is less than 3% of the axial resistance), $\Delta'_x = -54.90$ kips, or the lateral displacement of the foundation decreases six times. Thus, a completely misleading and much more alarming picture of the lateral foundation displacement is obtained if the lateral pile resistance is ignored, even when it is quite small.

c. Although the horizontal displacements of the foundation, Δ'_x, are always negative (that is, are always directed to the left), the vertical displacements

Δ'_y and the rotational movements α vary in direction, depending on the pile-soil conditions. The combination of these displacements sometimes results in most baffling values of the pile loads.

Thus, one would naturally expect that the greatest axial load is carried by pile 1. This is so in the medium and weak soils, but in the very weak soil, especially with the less rigid timber piles, pile 3 becomes the one which is loaded most. This condition is particularly pronounced in case of piles completely devoid of lateral resistance (case 1). Then the axial load on pile 3 is $P = 59.8$ kips, whereas on pile 1 it is only 23.5 kips. This striking result is accentuated by the negative axial load on the adjacent pile 4 and the positive load of 12.7 kips on pile 5.

In spite of this apparent irregularity of the pile loads in different groups the loads in the same group vary linearly. Thus, the loads on pile 2 are the mean between the load values on piles 1 and 3.

d. The greatest axial load when piles are endowed with lateral resistances in medium, weak, and even, in some cases, of very weak ground, varies between 35 kips and 39 kips, and it mostly acts on pile 1. This means that, when the lateral pile resistances are ignored (case 1), the greatest axial pile load is exaggerated by approximately 60%, and often a wrong pile is indicated as carrying the maximum load.

e. The constancy of the maximum axial load for nearly all pile-soil combinations is also surprising. Apart from cases 1, 2, and 4, involving very weak soil and flexible piles, the greatest variation of this load amounts to only 10%.

In comparing the more exact solutions based on all three pile ratios with the approximate ones involving the same values of r_1 and omitting the other ratios, the results on the whole are quite favorable to the approximate method. Both with respect to the deformations and to the pile loads, the two corresponding solutions in most cases are related in a definite manner, which may be predicted beforehand. In a few special cases this regularity is broken, evidently because of the peculiarities of the combinations of the pile ratios. These deviations are not serious, however, and the difference in the results by the two methods is nearly always quite small.

f. When the pile ratios r_2 and r_3 are ignored the horizontal displacement of the foundation is usually increased by an amount close to 10%, making the approximate method that much on the safe side. In one case of very weak soil (case 8) the approximate method is about 28% on the safe side. On the other hand, in the medium soils, and partly in the weak soils, the approximate method somewhat underestimates the horizontal displacements, which under these conditions are all comparatively small.

g. The values of the lateral pile loads, in nearly all cases, are found by the approximate method to be slightly higher than the ones found by the more exact method. In the two cases in which this relation is reversed, the difference between the two methods is very small, and may possibly be attributed to the errors of computation. The amount by which the lateral pile load found by

the approximate method exceeds the value found by the exact method is usually close to 10%, although in cases 8, 9, and 10 this excess varies between 15% and 30%. Except for cases 7 and 13, involving rigid piles in medium ground, all lateral pile loads are quite small.

h. The greatest axial loads are usually slightly greater when found by the approximate method, the difference varying between 0% and 10%. The reverse is true in five cases, in four of which the approximate method is on the unsafe side by from 2% to 4%, and in one case (case 9), by 10%.

From these remarks it may be seen that, on the whole, the approximate method compares favorably with the exact method within the limits of the example considered—that is, for the assumed arrangement of a five-pile bent, carrying the specified forces, under virtually all possible conditions of deformability of soil and piles. Even the extreme and rather rare discrepancy of 10% on the unsafe side and of 28% on the safe side should not be considered as excessive in a subject as uncertain as that of foundation problems. It is desirable, however, to verify this conclusion on the examples of some other pile arrangements and some other combinations of the external forces X, Y, and M.

Conclusions

The approximate method of analysis of pile loads presented in this paper has the following advantages:

1. It is reasonably simple and not too laborious.
2. It is free from the two common weaknesses of other current methods of analysis: Failure to investigate the horizontal foundation displacement, which is as important as the vertical displacement, and failure to make allowance for the lateral pile resistance, which leads to an enormous exaggeration of the lateral displacement accompanied sometimes by a drastic rearrangement of the pile loads, thus leading to an entirely erroneous picture of the mechanical behavior of the foundation.
3. The only physical property of piles and soil involved in the method is the ratio r_1, which can be determined simply as the ratio of the allowable lateral and axial pile loads, which data must be known to the designer, no matter what method of analysis he is prepared to use.
4. The error involved in disregarding the two minor pile ratios, r_2 and r_3, present in the more exact method, appears to be not excessive.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

PAPERS

PHOTOGRAMMETRIC SPACE RESECTION

BY CARL CHOW,[1] JUN. ASCE

SYNOPSIS

The principal object of this paper is to present a direct mathematical method of finding the elevation of the exposure station and consequently the tilt and the swing, if desired, of an aerial photograph. Since three points in space determine a plane (Fig. 1), the three vertical distances z_a, z_b, and z_c (Fig. 2), from a horizontal plane through the exposure station to the images, a, b, and c on the photograph of the three ground control points, A, B, and C, determine the position of the photograph in space. Therefore, the swing of the photograph is not necessarily required provided the values of z_a, z_b, and z_c have been specified. Furthermore, in the solution of the space resection problem only relative quantities are required expressive of the three ground points.

The problem of photogrammetric space resection has been solved in various ways.[2,3,4,5,6] The present paper is added to the list for several reasons. The method developed:

(1) Demonstrates an adaptation of vector analysis which is novel and interesting;
(2) Offers a more direct procedure than its predecessors, permitting the exact altitude of the exposure station in an aerial photograph;
(3) Provides a procedure of adapting photogrammetric work to areas where the customary type of geodetic control is impossible or infeasible;
(4) Yields all data necessary for the rectification of tilted photographs;
(5) Avoids the necessity of observing the swing;

NOTE.—Written comments are invited for immediate publication; to insure publication the last discussion should be submitted by **July 1, 1949.**

[1] Research Fellow, Army Map Service, Dept. of the Army, Washington, D. C.

[2] "Analytical Computations in Aerial Photogrammetry," by Earl Church, *Photogrammetric Engineering*, October, November, December, 1941, p. 212.

[3] "Manual of Photogrammetry," Am. Soc. of Photogrammetry, Pitman Pub. Corp., New York, N. Y., 1944.

[4] "Revised Geometry of the Aerial Photograph," by Earl Church, *Bulletin No. 15* in a series on aerial photogrammetry, Syracuse Univ., Syracuse, N. Y., June, 1945.

[5] "Applied Photogrammetry," by Ralph O. Anderson, Edwards Bros., Inc., Ann Arbor, Mich., 4th Ed., 1946, p. 391.

[6] "Space Resection Problems in Photogrammetry," by P. H. Underwood, *Transactions*, ASCE, Vol. 112, 1947, p. 921.

(6) Involves only simple computations and produces exceptionally accurate results; and

(7) Is a basic theory in the writer's plan to project his investigations of aerial photogrammetry into geodetic triangulation.

Section 1. Derivation of Fundamental Formulas

In Fig. 3, L indicates the position of the emergent nodal point of the lens of the aerial camera, and the length of Lo is equal to f, the focal length of the

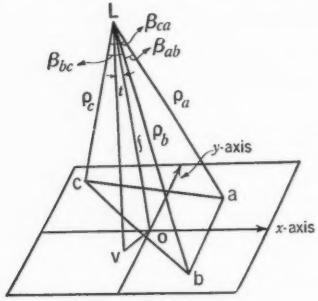


Fig. 1.—Plate Pyramid

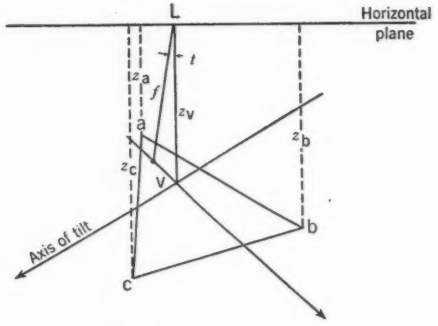


Fig. 2.—Vertical Relations

camera lens. Point L is referred to as the exposure station, and is used as the origin for the coordinate system. The coordinate axes, x and y, are the lines through L parallel to the geometric axes of the photograph, and the z-axis coincides with the lens axis.

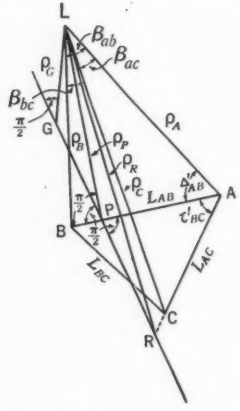


Fig. 3.—Ground Pyramid

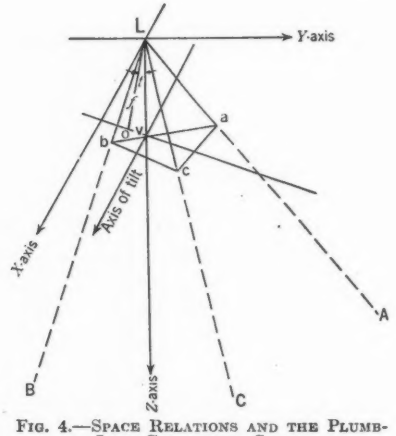


Fig. 4.—Space Relations and the Plumb-Line Coordinate System

If the plane coordinates of the images a, b, and c, measured on the photograph, are x_{pa},y_{pa}; x_{pb},y_{pb}; and x_{pc},y_{pc}, the space coordinates of the three images are x_{pa},y_{pa},f; x_{pb},y_{pb},f; and x_{pc},y_{pc},f. Then

$$\overline{La} = \rho_a = \sqrt{x^2_{pa} + y^2_{pa} + f^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1a)$$

$$\overline{Lb} = \rho_b = \sqrt{x^2_{pb} + y^2_{pb} + f^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1b)$$

and

$$\overline{Lc} = \rho_c = \sqrt{x^2_{pc} + y^2_{pc} + f^2} \quad \ldots\ldots (1c)$$

If these rays are regarded as vectors,

$$\overline{La} = \bar{\rho}_a, \quad x_{pa}\, i + y_{pa}\, j + f\, k \quad \ldots\ldots (2a)$$

$$\overline{Lb} = \bar{\rho}_b, \quad x_{pb}\, i + y_{pb}\, j + f\, k \quad \ldots\ldots (2b)$$

and

$$\overline{Lc} = \bar{\rho}_c, \quad x_{pc}\, i + y_{pc}\, j + f\, k \quad \ldots\ldots (2c)$$

in which i, j, and k are unit vectors referred to the corresponding axes in space. In accordance with the fundamental principles of vector analysis scalar products, $i.i = 1$, $j.j = 1$, $k.k = 1$, $i.j = j.k = k.i = 0$ give

$$\overline{La}.\overline{Lb} = \bar{\rho}_a.\bar{\rho}_b = x_{pa}\, x_{pb} + y_{pa}\, y_{pb} + f^2 = \rho_a\, \rho_b \cos \beta_{ab} \quad \ldots\ldots (3a)$$

$$\overline{Lb}.\overline{Lc} = \bar{\rho}_b.\bar{\rho}_c = x_{pb}\, x_{pc} + y_{pb}\, y_{pc} + f^2 = \rho_b\, \rho_c \cos \beta_{bc} \quad \ldots\ldots (3b)$$

and

$$\overline{Lc}.\overline{La} = \bar{\rho}_c.\bar{\rho}_a = x_{pc}\, x_{pa} + y_{pc}\, y_{pa} + f^2 = \rho_c\, \rho_a \cos \beta_{ca} \quad \ldots\ldots (3c)$$

in which the angles β are the face angles of the plate pyramid Labc. Therefore,

$$\cos \beta_{ab} = \frac{x_{pa}\, x_{pb} + y_{pa}\, y_{pb} + f^2}{\rho_a\, \rho_b} \quad \ldots\ldots (4a)$$

$$\cos \beta_{bc} = \frac{x_{pb}\, x_{pc} + y_{pb}\, y_{pc} + f^2}{\rho_b\, \rho_c} \quad \ldots\ldots (4b)$$

and

$$\cos \beta_{ca} = \frac{x_{pc}\, x_{pa} + y_{pc}\, y_{pa} + f^2}{\rho_c\, \rho_a} \quad \ldots\ldots (4c)$$

Now if the lengths ab, bc, and ca, Fig. 1, are designated by l_{ab}, l_{bc}, l_{ca}, respectively,

$$l^2_{ab} = (x_{pa} - x_{pb})^2 + (y_{pa} - y_{pb})^2 \quad \ldots\ldots (5a)$$

$$l^2_{bc} = (x_{pb} - x_{pc})^2 + (y_{pb} - y_{pc})^2 \quad \ldots\ldots (5b)$$

and

$$l^2_{ca} = (x_{pc} - x_{pa})^2 + (y_{pc} - y_{pa})^2 \quad \ldots\ldots (5c)$$

Then, if angle cab (Fig. 1) = τ_a, angle abc = τ_b, and angle bca = τ_c, vector operations identical to those leading to Eqs. 4 give

$$\cos \tau_a = \frac{(x_{pb} - x_{pa})\,(x_{pc} - x_{pa}) + (y_{pb} - y_{pa})\,(y_{pc} - y_{pa})}{l_{ab}\, l_{ca}} \quad \ldots\ldots (6a)$$

$$\cos \tau_b = \frac{(x_{pc} - x_{pb})\,(x_{pa} - x_{pb}) + (y_{pc} - y_{pb})\,(y_{pa} - y_{pb})}{l_{cb}\, l_{ab}} \quad \ldots\ldots (6b)$$

and

$$\cos \tau_c = \frac{(x_{pa} - x_{pc})\,(x_{pb} - x_{pc}) + (y_{pa} - y_{pc})\,(y_{pb} - y_{pc})}{l_{ca}\, l_{bc}} \quad \ldots\ldots (6c)$$

Similarly,

$$\cos \text{aob} = \cos \theta_1 = \frac{x_{pa}\, x_{pb} + y_{pa}\, y_{pb}}{r_a\, r_b} \quad \ldots\ldots (7a)$$

$$\cos \text{boc} = \cos \theta_2 = \frac{x_{pb}\, x_{pc} + y_{pb}\, y_{pc}}{r_b\, r_c} \quad \ldots\ldots (7b)$$

and

$$\cos \text{coa} = \cos \theta_3 = \frac{x_{pc}\, x_{pa} + y_{pc}\, y_{pa}}{r_c\, r_a} \quad \ldots\ldots\ldots\ldots\ldots\ldots (7c)$$

and

$$\cos \text{Lab} = \cos \Delta_{ab} = \frac{x_{pa}\,(x_{pb} - x_{pa}) + (y_{pb} - y_{pa})\, y_{pa}}{\rho_a\, l_{ab}} \quad \ldots\ldots\ldots (8a)$$

$$\cos \text{Lba} = \cos \Delta_{ba} = \frac{x_{pb}\,(x_{pa} - x_{pb}) + y_{pb}\,(y_{pa} - y_{pb})}{\rho_b\, l_{ab}} \quad \ldots\ldots\ldots (8b)$$

$$\cos \text{Lbc} = \cos \Delta_{bc} = \frac{x_{pb}\,(x_{pc} - x_{pb}) + y_{pb}\,(y_{pc} - y_{pb})}{\rho_b\, l_{bc}} \quad \ldots\ldots\ldots (8c)$$

$$\cos \text{Lcb} = \cos \Delta_{cb} = \frac{x_{pc}\,(x_{pb} - x_{pc}) + y_{pc}\,(y_{pb} - y_{pc})}{\rho_c\, l_{bc}} \quad \ldots\ldots\ldots (8d)$$

$$\cos \text{Lca} = \cos \Delta_{ca} = \frac{x_{pc}\,(x_{pa} - x_{pc}) + y_{pc}\,(y_{pa} - y_{pc})}{\rho_c\, l_{ca}} \quad \ldots\ldots\ldots (8e)$$

and

$$\cos \text{Lac} = \cos \Delta_{ac} = \frac{x_{pa}\,(x_{pc} - x_{pa}) + y_{pa}\,(y_{pc} - y_{pa})}{\rho_a\, l_{ca}} \quad \ldots\ldots\ldots (8f)$$

In Eqs. 6 and 7, r_a, r_b, and r_c represent the absolute lengths of oa, ob, and oc, respectively.

Section 2.—Determination of the Lengths of the Edges of the Ground Pyramid

The rays from the exposure station L to the ground points A, B, and C are designated ρ_A, ρ_B, and ρ_C; and the inclined, or natural ground, distances AB, BC, and CA, L_{AB}, L_{BC}, and L_{CA}, respectively. Then, by the law of cosines,

$$L^2_{AB} = \rho^2_A + \rho^2_B - 2\,\rho_A\,\rho_B \cos \beta_{ab} \quad \ldots\ldots\ldots\ldots\ldots (9a)$$

$$L^2_{BC} = \rho^2_B + \rho^2_C - 2\,\rho_B\,\rho_C \cos \beta_{bc} \quad \ldots\ldots\ldots\ldots\ldots (9b)$$

and

$$L^2_{CA} = \rho^2_C + \rho^2_A - 2\,\rho_C\,\rho_A \cos \beta_{ca} \quad \ldots\ldots\ldots\ldots\ldots (9c)$$

If three approximate values (ρ_A), (ρ_B), and (ρ_C) are substituted for ρ_A, ρ_B, and ρ_C, as given by

$$(\rho_A) = \frac{\rho_a\, L'_{AB}}{l_{ab}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (10a)$$

$$(\rho_B) = \frac{\rho_b\, L'_{BC}}{l_{bc}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (10b)$$

and

$$(\rho_C) = \frac{\rho_c\, L'_{CA}}{l_{ca}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (10c)$$

in which L'_{AB}, L'_{BC}, and L'_{CA} are horizontal distances of AB, BC, and CA,

respectively, then

$$L^2_{AB} = [(\rho_A) + \Delta\rho_A]^2 + [(\rho_B) + \Delta\rho_B]^2 - 2[(\rho_A) + \Delta\rho_A][(\rho_B) + \Delta\rho_B]\cos\beta_{ab} \quad (11a)$$

$$L^2_{BC} = [(\rho_B) + \Delta\rho_B]^2 + [(\rho_C) + \Delta\rho_C]^2 - 2[(\rho_B) + \Delta\rho_B][(\rho_C) + \Delta\rho_C]\cos\beta_{bc} \quad (11b)$$

and

$$L^2_{CA} = [(\rho_C) + \Delta\rho_C]^2 + [(\rho_A) + \Delta\rho_A]^2 - 2[(\rho_C) + \Delta\rho_C][(\rho_A) + \Delta\rho_A]\cos\beta_{ca} \quad (11c)$$

Let

$$L^2_{AB} - (\rho_A)^2 + (\rho_B)^2 - 2(\rho_A)(\rho_B)\cos\beta_{ab} = w_{AB} \quad (12a)$$

$$L^2_{BC} - (\rho_B)^2 + (\rho_C)^2 - 2(\rho_B)(\rho_C)\cos\beta_{bc} = w_{BC} \quad (12b)$$

and

$$L^2_{CA} - (\rho_C)^2 + (\rho_A)^2 - 2(\rho_C)(\rho_A)\cos\beta_{ca} = w_{CA} \quad (12c)$$

Omitting the second power terms in Eqs. 12,

$$2(\rho_A)\Delta\rho_A + 2(\rho_B)\Delta\rho_B - 2(\rho_A)\Delta\rho_B\cos\beta_{ab} - 2(\rho_B)\Delta\rho_A\cos\beta_{ab} = w_{AB} \quad (13a)$$

$$2(\rho_B)\Delta\rho_B + 2(\rho_C)\Delta\rho_C - 2(\rho_B)\Delta\rho_C\cos\beta_{bc} - 2(\rho_C)\Delta\rho_B\cos\beta_{bc} = w_{BC} \quad (13b)$$

and

$$2(\rho_C)\Delta\rho_C + 2(\rho_A)\Delta\rho_A - 2(\rho_C)\Delta\rho_A\cos\beta_{ca} - 2(\rho_A)\Delta\rho_C\cos\beta_{ca} = w_{CA} \quad (13c)$$

A solution of these three linear equations (Eqs. 13) yields $\Delta\rho_A$, $\Delta\rho_B$, and $\Delta\rho_C$. Then the second approximate values of ρ_A, ρ_B, and ρ_C will be $(\rho_A) + \Delta\rho_A$, $(\rho_B) + \Delta\rho_B$, and $(\rho_C) + \Delta\rho_C$, respectively. Repeated substitutions of these values in Eqs. 13 give accurate values of ρ_A, ρ_B, and ρ_C.

Section 3.—Determination of the Exposure Station Altitude

Fig. 3 shows a plane passed through L perpendicular to any side, say, AC, of the ground triangle ABC, cutting AC at the point R, cutting the side BC at point Q, and cutting the side AB at point P. This plane is also perpendicular to the plane ABC containing the three ground control points. By the law of sines,

$$\frac{\sin\Delta'_{AB}}{\rho_B} = \frac{\sin\beta_{ab}}{L_{AB}} \quad (14a)$$

$$\frac{\sin\Delta'_{BA}}{\rho_A} = \frac{\sin\beta_{ab}}{L_{AB}} \quad (14b)$$

$$\frac{\sin\Delta'_{BC}}{\rho_C} = \frac{\sin\beta_{bc}}{L_{BC}} \quad (14c)$$

$$\frac{\sin\Delta'_{CB}}{\rho_B} = \frac{\sin\beta_{bc}}{L_{BC}} \quad (14d)$$

$$\frac{\sin\Delta'_{CA}}{\rho_A} = \frac{\sin\beta_{ca}}{L_{CA}} \quad (14e)$$

and

$$\frac{\sin \Delta'_{AC}}{\rho_C} = \frac{\sin \beta_{ca}}{L_{CA}} \quad \ldots\ldots (14f)$$

in which the symbol Δ' refers to the face angles at the base of the ground pyramid L–ABC (Fig. 3). Eqs. 14 can each be transposed to express a value of $\sin \Delta'$.

Since the angles β are all measured by the photograph and computed as shown in Section 1 and since the inclined ground distances L_{AB}, L_{BC}, and L_{CA} are already known from the ground surveying, and ρ_A, ρ_B, and ρ_C have been found by the formulas in the preceding paragraph, the angles Δ'_{AB}, $\cdots$, Δ'_{BC} can be computed. Also the angles CAB $= \tau_A$, ABC $= \tau_B$, and BCA $= \tau_C$ in the plane of the triangle ABC can be computed by the three inclined ground distances L_{AB}, L_{BC}, and L_{CA} or by the following formulas, if the coordinates of the three ground points are known:

$$\cos \tau_A = \frac{(X_B - X_A)(X_C - X_A) + (Y_B - Y_A)(Y_C - Y_A) + (H_A - H_B)(H_A - H_C)}{L_{AB} L_{CA}} \quad \ldots (15a)$$

$$\cos \tau_B = \frac{(X_C - X_B)(X_A - X_B) + (Y_C - Y_B)(Y_A - Y_B) + (H_B - H_C)(H_B - H_A)}{L_{BC} L_{AB}} \quad \ldots (15b)$$

and

$$\cos \tau_C = \frac{(X_A - X_C)(X_B - Y_C) + (Y_A - Y_C)(Y_B - Y_C) + (H_C - H_A)(H_C - H_B)}{L_{CA} L_{BC}} \quad \ldots (15c)$$

in which X is the x-ordinate of a ground point; Y is the y-ordinate of a ground point; and H is the elevation of a ground point. By reference to Fig. 3,

$$\overline{\text{AR}} = \rho_A \cos \Delta'_{AC} \quad \ldots\ldots (16a)$$

$$\rho_R = \overline{\text{LR}} = \rho_A \sin \Delta'_{AC} \quad \ldots\ldots (16b)$$

$$\overline{\text{AP}} = \overline{\text{AR}} \sec \tau_A \quad \ldots\ldots (16c)$$

$$\overline{\text{PR}} = \overline{\text{AR}} \tan \tau_A \quad \ldots\ldots (16d)$$

$$\rho^2_P = \overline{\text{LP}^2} = \rho^2_A + \overline{\text{AP}^2} - 2\,\rho_A\,\overline{\text{AP}} \cos \Delta'_{AB} \quad \ldots\ldots (16e)$$

With ρ_R, ρ_P, and the length PR all determined, the area A_{pr} of the triangle LPR (Fig. 3) can be computed from the three sides. The usual formula in plane geometry is

$$A_{pr} = \sqrt{\Sigma\,(\Sigma - \rho_R)\,(\Sigma - \rho_P)\,(\Sigma - \overline{\text{PR}})} \quad \ldots\ldots (17a)$$

in which

$$\Sigma = \tfrac{1}{2}\,(\rho_R + \rho_P + \overline{\text{PR}}) \quad \ldots\ldots (17b)$$

Let a line LG (Fig. 3) be drawn from L perpendicular to PR, and let its length be designated ρ_G. Then, $A_{pr} = \frac{1}{2}\,\rho_G \times \overline{\text{PR}}$; or

$$\rho_G = \frac{2\,A_{pr}}{\overline{\text{PR}}} \quad \ldots\ldots (18)$$

which is the perpendicular distance from the exposure station L to the plane ABC containing the three ground control points. (It is not the elevation of the exposure station since the plane ABC is not at sea level, and since LG is not a vertical line.)

In addition to the use of Eq. 18, ρ_G also can be obtained as follows: Let V be the volume of the ground pyramid L–ABC, then[7]

$$6\,V = \rho_A\,\rho_B\,\rho_C \begin{vmatrix} 1 & \cos\beta_{ca} & \cos\beta_{bc} \\ \cos\beta_{ca} & 1 & \cos\beta_{ab} \\ \cos\beta_{bc} & \cos\beta_{ab} & 1 \end{vmatrix}^{\frac{1}{2}} = \rho_A\,\rho_B\,\rho_C\,\delta^{\frac{1}{2}} \quad \ldots\ldots (19a)$$

in which

$$\delta = 1 - \cos^2\beta_{ab} - \cos^2\beta_{bc} - \cos^2\beta_{ca} + 2\cos\beta_{ab}\cos\beta_{bc}\cos\beta_{ca} \quad \ldots (19b)$$

Also,

$$6\,V = \rho_G\,L_{AB}\,L_{AC}\sin\tau_A = \rho_G\,L_{AB}\,L_{BC}\sin\tau_B = \rho_G\,L_{BC}\,L_{AC}\sin\tau_C \quad \ldots (20)$$

Therefore,

$$\rho_G = \frac{\rho_A\,\rho_B\,\rho_C\,\delta^{\frac{1}{2}}}{L_{AB}\,L_{AC}\sin\tau_A} = \frac{\rho_A\,\rho_B\,\rho_C\,\delta^{\frac{1}{2}}}{L_{AB}\,L_{BC}\sin\tau_B} = \frac{\rho_A\,\rho_B\,\rho_C\,\delta^{\frac{1}{2}}}{L_{BC}\,L_{AC}\sin\tau_C} \quad \ldots\ldots (21)$$

Let $R_1 = \dfrac{\rho_A}{L_{AB}}$, $R_2 = \dfrac{\rho_B}{L_{BC}}$, and $R_3 = \dfrac{\rho_C}{L_{CA}}$,

$$\rho_G = R_1\,R_3\,\rho_B\,\delta^{\frac{1}{2}}/\sin\tau_A \quad \ldots\ldots (22a)$$

$$\rho_G = R_1\,R_2\,\rho_C\,\delta^{\frac{1}{2}}/\sin\tau_B \quad \ldots\ldots (22b)$$

and

$$\rho_G = R_2\,R_3\,\rho_A\,\delta^{\frac{1}{2}}/\sin\tau_C \quad \ldots\ldots (22c)$$

Next,

$$\overline{AG} = \sqrt{\rho^2_A - \rho^2_G} \quad \ldots\ldots (23a)$$

$$\overline{BG} = \sqrt{\rho^2_B - \rho^2_G} \quad \ldots\ldots (23b)$$

$$\overline{CG} = \sqrt{\rho^2_C - \rho^2_G} \quad \ldots\ldots (23c)$$

$$\overline{PG} = \sqrt{\rho^2_P - \rho^2_G} \quad \ldots\ldots (23d)$$

and

$$\overline{RG} = \sqrt{\rho^2_R - \rho^2_G} \quad \ldots\ldots (23e)$$

—all of which can be computed.

The elevation of point P (Fig. 3) can be found by extrapolation between H_A and H_B, the given elevations of points A and B, since both AB ($= L_{AB}$) and AP are now known. The elevation of point R can be found by extrapolation between the given elevations of points A and C, since both AC ($= L_{CA}$) and AR are now known. These calculated elevations of points P and R are called H_P and H_R, respectively. Finally the elevation of the point G (Fig. 3)

[7] "Applications of the Absolute Differential Calculus," by A. J. McConnell, Blackie and Son, Ltd. London, 1931, pp. 48–50.

can be found by interpolation between H_P and H_R, since the distances PG and RG are now both known.

The area of the triangle ABC in the inclined plane of the three control points is

$$A = \sqrt{\Sigma\,(\Sigma - L_{AB})\,(\Sigma - L_{BC})\,(\Sigma - L_{CA})} \quad \ldots\ldots (24a)$$

in which L_{AB}, L_{BC}, and L_{CA} are the inclined distances, and in which

$$\Sigma = \tfrac{1}{2}\,(L_{AB} + L_{BC} + L_{AC}) \quad \ldots\ldots (24b)$$

Also the area A' of the horizontal projection of the triangle ABC is

$$A' = \sqrt{\Sigma'\,(\Sigma' - L'_{AB})\,(\Sigma' - L'_{BC})\,(\Sigma' - L'_{CA})} \quad \ldots\ldots (25a)$$

in which L'_{AB}, L'_{BC}, and L'_{CA} are the horizontal distances, and where

$$\Sigma' = \tfrac{1}{2}\,(L'_{AB} + L'_{BC} + L'_{CA}) \quad \ldots\ldots (25b)$$

Then, if T represents the angle of inclination of the plane ABC to a horizontal plane,

$$\cos T = \frac{A'}{A} \quad \ldots\ldots (26)$$

This value of cos T can be found as follows: If the coordinates of the three ground control points are known as X_A,Y_A,H_A; X_B,Y_B,H_B; and X_C,Y_C,H_C, respectively, the equation of a plane which passes through the three ground control points is

$$\begin{vmatrix} X & Y & Z & 1 \\ X_A & Y_A & H_A & 1 \\ X_B & Y_B & H_B & 1 \\ X_C & Y_C & H_C & 1 \end{vmatrix} = 0 \quad \ldots\ldots (27)$$

Eq. 27 can be written in the form:

$$Z = f(X,Y) \quad \ldots\ldots (28)$$

since, then, by differential geometry,

$$\cos T = \frac{1}{\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}} \quad \ldots\ldots (29)$$

The altitude (Z) of the exposure station is then given by

$$Z = H_G + \rho_G \cos T \quad \ldots\ldots (30)$$

Section 4. Determination of the Tilt of the Photograph

In this section a space coordinate system is used with the origin at the exposure station L, a plumb line through L is designated as the Z-axis; a horizontal line through L parallel to the tilt axis of the photograph, as the X-axis; and a horizontal line through L perpendicular to this line, as the Y-axis—the last line being situated in the principal plane of the photograph (see Fig. 4).

Since the value of Z (the elevation of the exposure station) and values of ρ_a, ρ_b, ρ_c, ρ_A, ρ_B, and ρ_C have already been computed, and since the altitudes H_A, H_B, and H_C of the three ground points are given, the following relations—

$$z_a = \frac{\rho_a}{\rho_A}(Z - H_A) \quad \ldots\ldots (31a)$$

$$z_b = \frac{\rho_b}{\rho_B}(Z - H_B) \quad \ldots\ldots (31b)$$

and

$$z_c = \frac{\rho_c}{\rho_C}(Z - H_C) \quad \ldots\ldots (31c)$$

—can be used to compute z_a, z_b, and z_c, which are the z-coordinates of the images a, b, and c on the tilted photograph, referred to the coordinate system defined by Fig. 4.

The lengths l_{ab}, l_{bc}, and l_{ca} of the sides ab, bc, and ca, respectively, of the image triangle are known from the image coordinates measured on the photograph. The area a of this image triangle in the plane of the photograph can be found from

$$a = \sqrt{\sigma\,(\sigma - l_{ab})\,(\sigma - l_{bc})\,(\sigma - l_{ca})} \quad \ldots\ldots (32a)$$

in which

$$\sigma = \tfrac{1}{2}\,(l_{ab} + l_{bc} + l_{ca}) \quad \ldots\ldots (32b)$$

The horizontal projection of the sides ab, bc, and ca (l'_{ab}, l'_{bc}, l'_{ca}) can be found from

$$l'_{ab} = \sqrt{l^2_{ab} - (z_a - z_b)^2} \quad \ldots\ldots (33a)$$

$$l'_{bc} = \sqrt{l^2_{bc} - (z_b - z_c)^2} \quad \ldots\ldots (33b)$$

and

$$l'_{ca} = \sqrt{l^2_{ca} - (z_c - z_a)^2} \quad \ldots\ldots (33c)$$

Then the area a' of the horizontal projection of the image triangle in space is given by

$$a' = \sqrt{\sigma'\,(\sigma' - l'_{ab})\,(\sigma' - l'_{bc})\,(\sigma' - l'_{ca})} \quad \ldots\ldots (34a)$$

in which

$$\sigma' = \tfrac{1}{2}\,(l'_{ab} + l'_{bc} + l'_{ca}) \quad \ldots\ldots (34b)$$

Then the tilt of the photograph is given by

$$\cos t = \frac{a'}{a} \quad \ldots\ldots (35)$$

Section 5. Determination of Space Coordinates of the Ground Points and of Their Photographic Images, Referred to the Plumb-Line Coordinate System

Since the exposure station L is taken as the origin, its coordinates on the plumb-line coordinate system explained in the preceding section are O,O,O; and the ground points A, B, and C will have coordinates X_A,Y_A,$(Z - H_A)$; X_B, Y_B,$(Z - H_B)$; and X_C,Y_C,$(Z - H_C)$, respectively—H_A, H_B, and H_C being the elevations of these points. The horizontal coordinates are to be deter-

mined. By analytical geometry,

$$\rho^2_A = X^2_A + Y^2_A + (Z - H_A)^2 \quad \ldots\ldots (36a)$$

$$\rho^2_B = X^2_B + Y^2_B + (Z - H_B)^2 \quad \ldots\ldots (36b)$$

and

$$\rho^2_C = X^2_C + Y^2_C + (Z - H_C)^2 \quad \ldots\ldots (36c)$$

$$L^2_{AB} = (X_B - X_A)^2 + (Y_B - Y_A)^2 + (H_A - H_B)^2 \quad \ldots\ldots (37a)$$

$$L^2_{BC} = (X_C - X_B)^2 + (Y_C - Y_B)^2 + (H_B - H_C)^2 \quad \ldots\ldots (37b)$$

and

$$L^2_{CA} = (X_A - X_C)^2 + (Y_A - Y_C)^2 + (H_C - H_A)^2 \quad \ldots\ldots (37c)$$

Also, when ρ_A, ρ_B, and ρ_C are regarded as vectors,

$$\bar{\rho}_A, \quad X_A\, i + Y_A\, j + (Z - H_A)\, k \quad \ldots\ldots (38a)$$

$$\bar{\rho}_B, \quad X_B\, i + Y_B\, j + (Z - H_B)\, k \quad \ldots\ldots (38b)$$

and

$$\bar{\rho}_C, \quad X_C\, i + Y_C\, j + (Z - H_C)\, k \quad \ldots\ldots (38c)$$

The scalar products of pairs of these vectors are as follows:

$$\bar{\rho}_A \cdot \bar{\rho}_B = \rho_A\, \rho_B \cos \beta_{ab} = X_A\, X_B + Y_A\, Y_B + (Z - H_A)\, (Z - H_B) \quad \ldots (39a)$$

$$\bar{\rho}_B \cdot \bar{\rho}_C = \rho_B\, \rho_C \cos \beta_{bc} = X_B\, X_C + Y_B\, Y_C + (Z - H_B)\, (Z - H_C) \quad \ldots (39b)$$

and

$$\bar{\rho}_C \cdot \bar{\rho}_A = \rho_C\, \rho_A \cos \beta_{ca} = X_C\, X_A + Y_C\, Y_A + (Z - H_C)\, (Z - H_A) \quad \ldots (39c)$$

which, solved for the face angles β, yield

$$\cos \beta_{ab} = \frac{X_A\, X_B + Y_A\, Y_B + (Z - H_A)\, (Z - H_B)}{\rho_A\, \rho_B} \quad \ldots\ldots (40a)$$

$$\cos \beta_{bc} = \frac{X_B\, X_C + Y_B\, Y_C + (Z - H_B)\, (Z - H_C)}{\rho_B\, \rho_C} \quad \ldots\ldots (40b)$$

and

$$\cos \beta_{ca} = \frac{X_C\, X_A + Y_C\, Y_A + (Z - H_C)\, (Z - H_A)}{\rho_C\, \rho_A} \quad \ldots\ldots (40c)$$

Since Z is known, Eqs. 36 and 40 suffice for the determination of the six coordinates X_A, Y_A, X_B, Y_B, X_C, and Y_C.

Since t and z_a, z_b, and z_c are known in Section 4, the following equations will give y_a,y_b,y_c and x_a,x_b,x_c:

$$y_a = \frac{z_a - f \sec t}{\tan t} \quad \ldots\ldots (41a)$$

$$y_b = \frac{z_b - f \sec t}{\tan t} \quad \ldots\ldots (41b)$$

and

$$y_c = \frac{z_c - f \sec t}{\tan t} \quad \ldots\ldots (41c)$$

and

$$x_a = \pm (\rho^2_a - y^2_a - z^2_a)^{\frac{1}{2}} \quad \ldots\ldots (42a)$$

$$x_b = \pm (\rho^2_b - y^2_b - z^2_b)^{\frac{1}{2}} \quad \ldots\ldots (42b)$$

$$x_c = \pm (\rho^2_c - y^2_c - z^2_c)^{\frac{1}{2}} \quad \ldots\ldots (42c)$$

The signs of x_a, x_b, and x_c are selected according to the positions of the images of a, b, and c, with respect to the intersection lines of the XZ-plane and the YZ-plane with the plane of the photograph. These intersection lines can be drawn as follows:

By the lengths of l_1, l_2, and l_3, the axis of tilt (intersection line of the XZ-plane) can be determined, thus:

$$\frac{z_v - z_b}{z_a - z_b} = \frac{l_1}{l_{ab}} \quad \ldots\ldots (43a)$$

$$\frac{z_v - z_c}{z_b - z_c} = \frac{l_2}{l_{bc}} \quad \ldots\ldots (43b)$$

and

$$\frac{z_v - z_a}{z_c - z_a} = \frac{l_3}{l_{ac}} \quad \ldots\ldots (43c)$$

in which

$$z_v = \overline{\mathrm{Lv}} = \sqrt{f^2 + f^2 \tan^2 t} = f \sec t. \quad \ldots\ldots (43d)$$

A line drawn through o and perpendicular to the axis of tilt is the intersection line of YZ-plane and its direction is taken from o to v as positive. The direction of the axis of tilt is taken according to the right-hand rule from the x-axis to the y-axis.

Since x_a,y_a,z_a; x_b,y_b,z_b; and x_c,y_c,z_c and their signs are known, then

$$X'_A = \frac{\rho_A}{\rho_a} x'_a; \qquad Y'_A = \frac{\rho_A}{\rho_a} y'_a \quad \ldots\ldots (44a)$$

$$X'_B = \frac{\rho_B}{\rho_b} x'_b; \qquad Y'_B = \frac{\rho_B}{\rho_b} y'_b \quad \ldots\ldots (44b)$$

and

$$X'_C = \frac{\rho_C}{\rho_c} x'_c; \qquad Y'_C = \frac{\rho_C}{\rho_c} y'_c \quad \ldots\ldots (44c)$$

can be calculated. Substituting these values from Eq. 44a to Eq. 44c into Eqs. 36 and 39 or 40, the latter equations should be satisfied. However, because of the inaccuracy of cos t and the inaccuracy of computing large values from small values, the residuals from the substitutions are unavoidable. The accurate values of X'_A,Y'_A; X'_B,Y'_B; and X'_C,Y'_C will be obtained, as follows:

$$\rho^2_A - (Z - H_A)^2 = (k_1) \quad \ldots\ldots (45a)$$

$$\rho^2_B - (Z - H_B)^2 = (k_2) \quad \ldots\ldots (45b)$$

and

$$\rho^2_C - (Z - H_C)^2 = (k_3) \quad \ldots\ldots (45c)$$

$$\rho_A \rho_B \cos \beta_{ab} - (Z - H_A)(Z - H_B) = (k_4) \quad \ldots\ldots (46a)$$

$$\rho_B \rho_C \cos \beta_{bc} - (Z - H_B)(Z - H_C) = (k_5) \quad \ldots\ldots (46b)$$

and

$$\rho_C \rho_A \cos \beta_{ca} - (Z - H_C)(Z - H_A) = (k_6) \quad \text{(46c)}$$

Then

$$(k_1) = (X'_A + \Delta X_A)^2 + (Y'_A + \Delta Y_A)^2 \quad \text{(47a)}$$

$$(k_2) = (X'_B + \Delta X_B)^2 + (Y'_B + \Delta Y_B)^2 \quad \text{(47b)}$$

$$(k_3) = (X'_C + \Delta X_C)^2 + (Y'_C + \Delta Y_C)^2 \quad \text{(47c)}$$

$$(k_4) = (X'_A + \Delta X_A)(X'_B + \Delta X_B) + (Y'_A + \Delta Y_A)(Y'_B + \Delta Y_B) \quad \text{(47d)}$$

$$(k_5) = (X'_B + \Delta X_B)(X'_C + \Delta X_C) + (Y'_B + \Delta Y_B)(Y'_C + \Delta Y_C) \quad \text{(47e)}$$

and

$$(k_6) = (X'_C + \Delta X_C)(X'_A + \Delta X_A) + (Y'_C + \Delta Y_C)(Y'_A + \Delta Y_A) \quad \text{(47f)}$$

Let the second power terms of ΔX and ΔY be omitted and

$$(k_1) - [(X'_A)^2 + (Y'_A)^2] = (l_1) \quad \text{(48a)}$$

$$(k_2) - [(X'_B)^2 + (Y'_B)^2] = (l_2) \quad \text{(48b)}$$

$$(k_3) - [(X'_C)^2 + (Y'_C)^2] = (l_3) \quad \text{(48c)}$$

$$(k_4) - (X'_A X'_B + Y'_A Y'_B) = (l_4) \quad \text{(48d)}$$

$$(k_5) - (X'_B X'_C + Y'_B Y'_C) = (l_5) \quad \text{(48e)}$$

and

$$(k_6) - (X'_C X'_A + Y'_C Y'_A) = (l_6) \quad \text{(48f)}$$

Then Eqs. 47 become

$$(l_1) = 2 X'_A \Delta X_A + 2 Y'_A \Delta Y_A \quad \text{(49a)}$$

$$(l_2) = 2 X'_B \Delta X_B + 2 Y'_B \Delta Y_B \quad \text{(49b)}$$

$$(l_3) = 2 X'_C \Delta X_C + 2 Y'_C \Delta Y_C \quad \text{(49c)}$$

$$(l_4) = X'_A \Delta X_B + X'_B \Delta X_A + Y'_A \Delta Y_B + Y'_B \Delta Y_A \quad \text{(49d)}$$

$$(l_5) = X'_B \Delta X_C + X'_C \Delta X_B + Y'_B \Delta Y_C + Y'_C \Delta Y_B \quad \text{(49e)}$$

and

$$(l_6) = X'_C \Delta X_A + X'_A \Delta X_C + Y'_C \Delta Y_A + Y'_A \Delta Y_C \quad \text{(49f)}$$

A solution of Eqs. 49 gives ΔX_A, ΔY_A, ΔX_B, ΔY_B, ΔX_C, and ΔY_C. Then the second approximate values of X_A, Y_A, X_B, Y_B, X_C, and Y_C will be $X'_A + \Delta X_A$, $Y'_A + \Delta Y_A$, $X'_B + \Delta X_B$, $Y'_B + \Delta Y_B$, $X'_C + \Delta X_C$, and $Y'_C + \Delta Y_C$, respectively. Usually the second approximate values are accurate enough; if not, the same procedure can be repeated until the designer is satisfied.

After X_A, Y_A, X_B, Y_B, X_C, Y_C, and Z have been found, the accurate coordinates of the corresponding images can be computed directly by the following relations:

$$\frac{x_a}{X_A} = \frac{y_a}{Y_A} = \frac{z_a}{Z - H_A} = \frac{\rho_a}{\rho_A} \quad \text{(50a)}$$

$$\frac{x_b}{X_B} = \frac{y_b}{Y_B} = \frac{z_b}{Z - H_B} = \frac{\rho_b}{\rho_B} \quad \text{(50b)}$$

and

$$\frac{x_c}{X_C} = \frac{y_c}{Y_C} = \frac{z_c}{Z - H_C} = \frac{\rho_c}{\rho_C} \quad \ldots\ldots (50c)$$

From Fig. 2,

$$z_a = y_a \tan t + f \sec t \quad \ldots\ldots (51a)$$

$$z_b = y_b \tan t + f \sec t \quad \ldots\ldots (51b)$$

and

$$z_c = y_c \tan t + f \sec t \quad \ldots\ldots (51c)$$

From any two of Eqs. 51,

$$z_a - z_b = (y_a - y_b) \tan t \quad \ldots\ldots (52a)$$

$$z_b - z_c = (y_b - y_c) \tan t \quad \ldots\ldots (52b)$$

and

$$z_c - z_a = (y_c - y_a) \tan t \quad \ldots\ldots (52c)$$

Hence,

$$\tan t = \frac{z_a - z_b}{y_a - y_b} = \frac{z_b - z_c}{y_b - y_c} = \frac{z_c - z_a}{y_c - y_a} \quad \ldots\ldots (53)$$

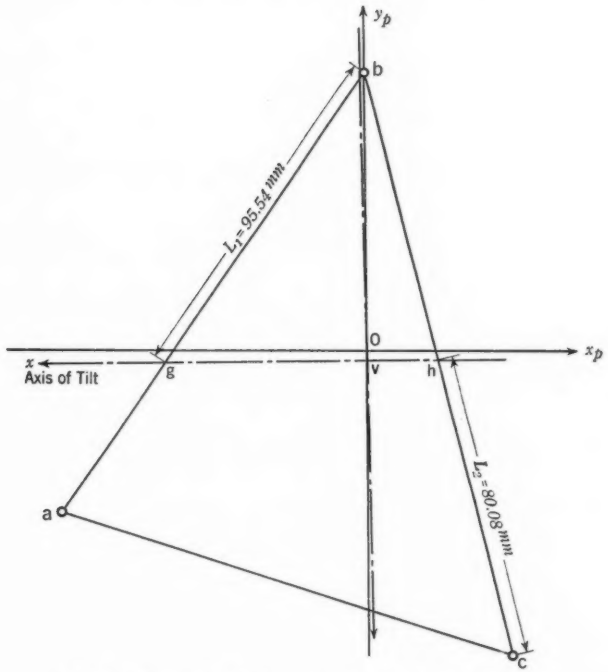


FIG. 5.—GRAPHICAL DETERMINATION OF THE AXIS OF TILT

Theoretically the three values of tan t in Eq. 53 should be identical. Actually, exact checks are seldom obtained, especially when the tilts approach zero, because the differences between pairs of z-values are so small that the tangents become very sensitive to small errors therein. However, the average of the three values of tilt will be very accurate.

Since t is known—

$$\overline{Ov} = f \tan t \quad \ldots\ldots (54)$$

—gives the distance from the principal point to nadir point on the principal line (see Fig. 5). Now suppose the vectors of the rays from the exposure station to the three image points and the principal point are as follows:

$$\overline{La} = \bar{\rho}_a, \qquad x_a\, i + y_a\, j + z_a\, k \quad \ldots\ldots (55a)$$

$$\overline{Lb} = \bar{\rho}_b, \qquad x_b\, i + y_b\, j + z_b\, k \quad \ldots\ldots (55b)$$

$$\overline{Lc} = \bar{\rho}_c, \qquad x_c\, i + y_c\, j + z_c\, k \quad \ldots\ldots (55c)$$

and

$$\overline{Lo} = \bar{\rho}_o, \qquad 0 + y\, j + z\, k \quad \ldots\ldots (55d)$$

in which x, y, and z represent the unknown coordinates of the principal point referred to the current plumb-line coordinate system. Taking scalar products,

$$\bar{\rho}_a \cdot \bar{\rho}_o = f^2 = 0 + y_a\, y + z_a\, z \quad \ldots\ldots (56a)$$

$$\bar{\rho}_b \cdot \bar{\rho}_o = f^2 = 0 + y_b\, y + z_b\, z \quad \ldots\ldots (56b)$$

and

$$\bar{\rho}_c \cdot \bar{\rho}_o = f^2 = 0 + y_c\, y + z_c\, z \quad \ldots\ldots (56c)$$

TABLE 1.—NUMERICAL EXAMPLE; PHYSICAL DATA FOR

Point[a] (see Fig. 3)	Photo Coordinates		Ground Coordinates			Δx	Δy	l	ρ
	x	y	X	Y	Z				
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
a_2 a	−81.92	−43.59	+ 5,000	+10,000	1,800	+ 81.92	+118.41	143.985	176.383
c_5 b	0.00	+74.82	+15,000	+25,000	1,000	+ 39.22	−155.89	160.748	167.625
d_1 c	+39.22	−81.07	+20,000	+ 5,000	900	−121.14	+ 37.48	126.806	174.959

[a] In Col. 1, a_2, c_5, and d_1 are the original symbols for the three particular images, made by Earl Church.

If Eqs. 56 are solved for x, y, and z,

$$x = 0 \quad \ldots\ldots (57a)$$

$$y = f^2 \begin{vmatrix} 1 & z_a \\ 1 & z_b \end{vmatrix} \Big/ \begin{vmatrix} y_a & z_a \\ y_b & z_b \end{vmatrix} \quad \ldots\ldots (57b)$$

and

$$z = f^2 \begin{vmatrix} y_a & 1 \\ y_b & 1 \end{vmatrix} \Big/ \begin{vmatrix} y_a & z_a \\ y_b & z_b \end{vmatrix} \quad \ldots\ldots (57c)$$

If it is desirable to utilize the photographic axes to locate the nadir point, the following procedure can be adopted. Let the coordinates of the nadir point be the unknown quantities x_{pv}, and y_{pv}, and let the known quantity $z_{pv} = f$,

be referred to the photographic coordinate system; thus (compare Eqs. 55),

$$\bar{\rho}_a, \quad x_{pa}\, i + y_{pa}\, j + f\, k \ldots\ldots\ldots\ldots\ldots\ldots (58a)$$

$$\bar{\rho}_b, \quad x_{pb}\, i + y_{pb}\, j + f\, k \ldots\ldots\ldots\ldots\ldots\ldots (58b)$$

$$\bar{\rho}_c, \quad x_{pc}\, i + y_{pc}\, j + f\, k \ldots\ldots\ldots\ldots\ldots\ldots (58c)$$

and

$$\bar{\rho}_v, \quad x_{pv}\, i + y_{pv}\, j + f\, k \ldots\ldots\ldots\ldots\ldots\ldots (58d)$$

Taking scalar products (compare Eqs. 56).

$$\bar{\rho}_a \cdot \bar{\rho}_v = \rho_a\, \rho_v \cos V_a = x_{pa}\, x_{pv} + y_{pa}\, y_{pv} + f^2 \ldots\ldots\ldots (59a)$$

$$\bar{\rho}_b \cdot \bar{\rho}_v = \rho_b\, \rho_v \cos V_b = x_{pb}\, x_{pv} + y_{pb}\, y_{pv} + f^2 \ldots\ldots\ldots (59b)$$

and

$$\bar{\rho}_c \cdot \bar{\rho}_v = \rho_c\, \rho_v \cos V_c = x_{pc}\, x_{pv} + y_{pc}\, y_{pv} + f^2 \ldots\ldots\ldots (59c)$$

in which V is the angle between ρ and the Z-axis.

In Eqs. 59, since Z is known, values of V van be computed by

$$\cos V_a = \frac{Z - H_A}{\rho_A} \ldots\ldots\ldots\ldots\ldots\ldots (60a)$$

$$\cos V_b = \frac{Z - H_B}{\rho_B} \ldots\ldots\ldots\ldots\ldots\ldots (60b)$$

E DETERMINATION OF ELEVATION Z AND ANGLE OF TILT

ΔX	ΔY	ΔH	Horizontal		Inclined		Point
			$(L')^2$	L'	L^2	L	
(11)	(12)	(13)	(14)	(15)	(16)	(17)	(1)
-10,000	+15,000	−800	325,000,000	L'_{AB} = 18,027.756	325,640,000	L'_{AB} = 18,045.489	a_2 a
- 5,000	−20,000	−100	425,000,000	L'_{BC} = 20,615.528	425,010,000	L'_{BC} = 20,615.771	c_5 b
-15,000	+ 5,000	+900	250,000,000	L'_{CA} = 15,811.388	250,810,000	L'_{CA} = 15,836.982	d_1 c

ρ: 176.383, 167.625, 174.959 (10)

ssoc. M. ASCE. In this paper, these three points are regarded as a, b, and c, respectively.

and

$$\cos V_c = \frac{Z - H_C}{\rho_C} \ldots\ldots\ldots\ldots\ldots\ldots (60c)$$

From Eqs. 59 one can solve directly for ρ_v, x_{pv}, and y_{pv}. By the values of x_{pv} and y_{pv}, the distance Ov (Fig. 5), and consequently the tangent of the angle of tilt, can be computed.

Numerical Example.—To demonstrate the theory recommended herein it is proposed to test it against a standard problem that has been solved elsewhere.[8] The focal distance f is given as 150 mm and the ordinates and other physical data are presented in Table 1. With these data, Eqs. 3 can be solved and the

[8] "Manual of Photogrammetry," Am. Soc. of Photogrammetry, Pitman Pub. Corp., New York, N. Y., 1944, p. 558.

results entered in Table 2 as shown; and Eqs. 4 then yield the values of $\cos \beta$ in line 1, Table 3.

As a first approximation, Eqs. 10 are solved to yield: $\rho'_A = 22{,}084$; $\rho'_B = 21{,}497$; and $\rho'_C = 21{,}816$. Substituting these approximate values of ρ in Eqs. 9, successive approximation yields:

Distance	ρ	ρ^2
ρ_A	21,532	463,627,024
ρ_B	21,646	468,549,316
ρ_C	22,305	497,513,025

These values, with the aid of Table 3, then permit the solution of Eqs. 14, as follows:

Angles Δ'	Sine	Cosine
Δ'_{AB}	+0.910 870	+0.413 141
Δ'_{AC}	+0.948 072	+0.318 054

TABLE 2.—COMPUTATION OF FACE ANGLES β, FIG. 1

Eq. No.	First quantity	+	Second quantity	+	f^2	=	$f(\rho) \cos \beta$
3*a*	0.000		−3.261.404		+22,500.000		+19,238.596
3*b*	0.000		−6.065.657		+22,500.000		+16,434.343
3*c*	−3,212.902		+3,533.841		+22,500.000		+22,820.939

TABLE 3.—NATURAL SINES AND COSINES OF FACE ANGLES β, FOR SOLUTION OF EQS. 14

Line	Function	β_{ab}	β_{bc}	β_{ca}
1	cos	0.650 6956	0.560 3731	0.739 5039
2	cos²	0.423 4048	0.314 0180	0.546 8690
3	sin²	0.576 5952	0.685 9820	0.453 1310
4	sin	0.759 339	0.828 240	0.673 150

TABLE 4.—SOLUTION OF EQS. 15

Line	(X_B-X_A)	×	(X_C-X_A)	+	(Y_B-Y_A)	×	(Y_C-Y_A)	+	(H_A-H_B)	×	(H_A-H_C)	=	$L_{AB} L_{CA} \times \cos \tau_A$
1	15,000		20,000		25,000		5,000		1,800		1,800		....
2	5,000		5,000		10,000		10,000		1,000		900		....
3	+10,000		+15,000		+15,000		−5,000		+800		+900		....
4	+ 150,000,000				− 75,000,000				+ 720,000				....
5	Σ = 75,720,000												$L_{AB} L_{CA} \times$ (+0.264 95296)

Likewise (see Fig. 3), $\rho_P = \rho_A \sin \Delta'_{AB} = 19{,}613$; and $\overline{AR} = \rho_A \cos \Delta'_{AB} = 8{,}896$. A tabular solution of Eq. 15*a* is demonstrated by Table 4 which reveals that $\cos \tau_A = +0.264\,95296$, and therefore $\tan \tau_A = 3.639\,368$. Again, from Fig. 3, $\overline{PR} = \overline{AR} \tan \tau_A = 32{,}376$; and $\overline{AP} = + \overline{AR} \sec \tau_A$

= 33,576. Similar to Eq. 16*c*, then, $\rho^2_R = \rho^2_A + \overline{AR}^2 - 2\,\rho_A\,\overline{AR}\cos\Delta'_{AC}$; that is, $\rho^2_R = 1{,}131{,}095{,}158$ and $\rho_R = 33{,}632$. Similar to Eq. 17*b*, $\Sigma = \frac{1}{2}(\rho_P + \rho_R + \overline{PR}) = 42{,}810$. For substitution in Eq. 17*a*, $\Sigma - \rho_P = 23{,}197$; $\Sigma - \rho_R = 9{,}178$; and $\Sigma - \overline{PR} = 10{,}434$. With this value of A, Eq. 18 yields $\rho_G = 19{,}050$. Other quantities needed in the solution are

$$(a) \qquad \frac{H_A - H_C}{L_{CA}} = \frac{900}{15{,}837} = 0.056\,8289$$

$$(b) \qquad \frac{H_A - H_B}{L_{AB}} = \frac{800}{18{,}046} = 0.044\,3324$$

$$(c) \qquad \Delta H_R = (a) \times 33{,}576 = 1{,}908$$

$$(d) \qquad \Delta H_P = (b) \times 8{,}896 = 398$$

$$(e) \qquad H_R = H_A - (c) = 1{,}800 - 1{,}908 = -\,108$$

$$(f) \qquad H_P = H_A - (d) = 1{,}800 - 398 = +\,1{,}406$$

$$(g) \qquad \overline{PG}\ \text{(Fig. 3)} = (\rho^2_P - \rho^2_G)^{\frac{1}{2}} = 4{,}666$$

$$(h) \qquad H_R - H_P = -\,(e) - (f) = -\,1{,}514$$

$$(i) \qquad \frac{H_R - H_P}{\overline{PR}} = \frac{(h)}{32{,}376} = -\,0.046\,763\,034$$

$$(j) \qquad H_G = (f) + (i) \times (g) = 1{,}188$$

The equation of a plane that passes through the three ground control points is

$$X\begin{vmatrix} 0 & 0 & 1 \\ 15{,}000 & -800 & 0 \\ -20{,}000 & -100 & 0 \end{vmatrix} - Y\begin{vmatrix} 0 & 0 & 1 \\ 10{,}000 & -800 & 0 \\ 5{,}000 & -100 & 0 \end{vmatrix} + Z\begin{vmatrix} 0 & 0 & 1 \\ 10{,}000 & 15{,}000 & 0 \\ 5{,}000 & -20{,}000 & 0 \end{vmatrix} = 0$$

In Eq. 30,

$$\frac{\partial Z}{\partial X} = +\,0.063\,6363636; \qquad \frac{\partial Z}{\partial Y} = +\,0.010\,9090909;$$

$$\left(\frac{\partial Z}{\partial Y}\right)^2 = +\,0.000\,1190083; \qquad \text{and} \qquad \left(\frac{\partial Z}{\partial X}\right)^2 = +\,0.004\,0495868.$$

TABLE 5.—Computation of Horizontal Sides l'_{ab}, l'_{bc}, and l'_{ca}

Line	Symbol	Side ab	Side bc	Side ca
1	$Z - h$	$Z - H_A = 18{,}398$	$Z - H_B = 19{,}198$	$Z - H_C = 19{,}298$
2	z	$z_a = 150.713$	$z_b = 148.668$	$z_c = 151.372$
3	l^2	$l^2_{ab} = 20{,}731.6802$	$l^2_{bc} = 25{,}839.9195$	$l^2_{ca} = 16{,}079.7616$
4	Δz^2	$(z_a - z_b)^2 = 4.1698$	$(z_c - z_b)^2 = 7.3116$	$(z_c - z_a)^2 = 0.4382$
5	$(l')^2$	$(l'_{ab})^2 = 20{,}727.5104$	$(l'_{bc})^2 = 25{,}832.6079$	$(l'_{ca})^2 = 16{,}079.3234$
6	l'	$l'_{ab} = 143.9705$	$l'_{bc} = 160.7252$	$l'_{ca} = 126.8043$

Therefore, $\cos T = 0.997\,9222$; and $\rho_G \cos T = 19{,}010$. Eq. 30 then yields $Z = 19{,}010 + 1{,}188 = 20{,}198$ ft.

Determination of the Tilt and the Intersection Lines of the ZX-Plane and the ZY-Plane with the Photograph.—From data compiled in Table 5, values of σ' in

Eqs. 34 (and corresponding values of σ') are determined as follows:

Symbol	Horizontal projection (symbols primed)	Inclined plane
l_{bc}	160.725	160.748
l_{ca}	126.804	126.806
l_{ab}	143.971	143.985
2σ	431.500	431.539
σ	215.750	215.7695
$\sigma - l_{ab}$	71.779	71.785
$\sigma - l_{bc}$	55.025	55.022
$\sigma - l_{ca}$	88.946	88.965

TABLE 6.—SOLUTION OF EQ. 53

Quantity	Point A (or a)	Point B (or b)	Point C (or c)
X	$X_A = +9{,}987$	$X_B = -78$	$X_C = -4{,}946$
Y	$Y_A = +5{,}040$	$Y_B = -9{,}999$	$Y_C = +10{,}032$
x	$x_a = +81.81$	$x_b = -0.60$	$x_c = -38.80$
y	$y_a = \frac{176.383 \times 5{,}040}{21{,}532} = +41.29$	$y_b = \frac{167.625 \times (-9{,}999)}{21{,}640} = -77.45$	$y_c = \frac{174.959 \times 10{,}032}{22{,}305} = +78.69$
Δy	$y_a - y_b = 118.74$	$y_b - y_c = 156.14$	$y_c - y_a = 37.40$
Δz	$z_a - z_b = 2.0423$	$z_b - z_c = 2.704$	$z_c - z_a = 0.6617$
Tan t	0.017 19976	0.017 31779	0.017 69251

TABLE 7.—SOLUTION OF EQS. 59 AND 60 TO DETERMINE THE NADIR POINT V

Point	x	y	ρ (photo pyramid)	$Z - H$	ρ (ground pyramid)	cos V	$\rho \cos V$	
a	−81.92	−43.59	176.383	18,398	21,532	0.854 4492	150.7103	$\frac{x_{pa}}{\rho_a \cos}$
b	0.00	+74.82	167.625	19,198	21,646	0.886 9075	148.6679	$\frac{x_{pb}}{\rho_b \cos}$
c	+39.22	−81.07	174.959	19,298	22,305	0.865 1872	151.3723	$\frac{x_{pc}}{\rho_c \cos}$

By Eq. 3, $\cos t = a'/a = 0.999\,8335$; and $t = 1° 02.5'$. Since $z_v = f \sec t$, $z_v = 150.02498$ and $\overline{Ov} = f \tan t = 150 \times 0.0181 = 2.715$ mm. By geometry (see Figs. 2 and 5),

$$\frac{z_v - z_b}{z_a - z_b} = \frac{l_1}{l_{ab}} = \frac{1.357}{2.045} = \frac{l_1}{143.985} \quad \ldots\ldots (61a)$$

$$\frac{z_c - z_v}{z_c - z_b} = \frac{l_2}{l_{bc}} = \frac{1.347}{2.704} = \frac{l_2}{160.748} \quad \ldots\ldots (61b)$$

and

$$\frac{z_c - z_v}{z_c - z_a} = \frac{l_3}{l_{ca}} = \frac{1.347}{0.659} = \frac{l_3}{126.806} \quad \ldots\ldots (61c)$$

Eqs. 61 each contain the "ingredients" for determining the values of l, in millimeters, respectively, as follows (see Fig. 5): $l_1 = \overline{bg} = 95.545$; $l_2 = \overline{ch}$

$= 80.077$; and $l_3 = 259.192$. Values of $\tan t$, computed by Eq. 53, are given in the last line of Table 6. The average of the three values is 0.017 40302, from which a value of $t = 0° 59.8'$ is computed. Computations for the determination of the nadir point with respect to the photograph coordinate system (see Fig. 5) are listed in Table 7. From these data, two equations can be set up for simultaneous solution (by Eqs. 59):

$$-0.5436\,x_{pv} - 0.79250\,y_{pv} - 2.0510 = 0 \quad \ldots\ldots\ldots\ldots (62a)$$

and

$$-0.25910\,x_{pv} + 1.03884\,y_{pv} + 2.7039 = 0 \quad \ldots\ldots\ldots\ldots (62b)$$

—from which $x_{pv} = +0.016$ and $y_{pv} = -2.600$. Since $\overline{Ov} = 2.6000$, Eq. 54 yields, $\tan t = 2.600/150 = 0.0173333$; and $t = 0° 59.6'$. Likewise, $\tan s = -0.006\,15385$; and $s = 179° 39.0'$.

Section 6. Air Base and Space Intersection

Of two aerial photographs, taken successively, the first can be analyzed by the procedure demonstrated in Section 5. The second photograph, containing a different exposure station (point 2, Fig. 6), can also be analyzed by that procedure if it contains images of three control points that appear in the first photograph. In other words, the elevation of the second exposure station (point L_2, Fig. 6) can be determined, as well as the space coordinates (X'_A,

INT WITH RESPECT TO THE PHOTOGRAPH COORDINATE SYSTEM

Numerical Coefficients in Eqs. 59			Point
$f(x)$	$f(y)$	$f(f)$	
$\frac{x_{pa}}{\rho_a \cos V_a} = -0.54356$	$\frac{y_{pa}}{\rho_a \cos V_a} = -0.28923$	$\frac{f^2}{\rho_a \cos V_a} = +149.2930$	a
$\frac{x_{pb}}{\rho_b \cos V_b} = +0.00000$	$\frac{y_{pb}}{\rho_b \cos V_b} = +0.50327$	$\frac{f^2}{\rho_b \cos V_b} = +151.2440$	b
$\frac{x_{pc}}{\rho_c \cos V_c} = +0.25910$	$\frac{y_{pc}}{\rho_c \cos V_c} = -0.53557$	$\frac{f^2}{\rho_c \cos V_c} = +148.6401$	c

Y'_A, X'_B, Y'_B, X'_C, and Y'_C) of the three ground points (A, B, and C, Fig. 6) referred to the second plumb-line coordinate system.

Air Base.—In Fig. 6(*a*), the images of points A, B, and C are visible in two photographs taken in sequence. It is possible to compute the space ordinates (X_2 and Y_2) of the second exposure station, with respect to the first plumb-line coordinate system, by utilizing the scalar products of the second set of vectors (ρ'_A, ρ'_B, and ρ'_C). Thus (compare Eqs. 38):

$$\rho'_A: \quad (X_A - X_2)\,i + (Y_A - Y_2)\,j + (Z_2 - H_A)\,k \quad \ldots\ldots (63a)$$

$$\rho'_B: \quad (X_B - X_2)\,i + (Y_B - Y_2)\,j + (Z_2 - H_B)\,k \quad \ldots\ldots (63b)$$

and

$$\rho'_C: \quad (X_C - X_2)\,i + (Y_C - Y_2)\,j + (Z_2 - H_C)\,k \quad \ldots\ldots (63c)$$

The scalar products of pairs of these vectors are

$$\rho'_A \rho'_B \cos \beta'_{ab} = (X_A - X_2)(X_B - X_2) + (Y_A - Y_2)(Y_B - Y_2) + (Z_2 - H_A)(Z_2 - H_B) \quad (64a)$$

$$\rho'_B \rho'_C \cos \beta'_{bc} = (X_B - X_2)(X_C - X_2) + (Y_B - Y_2)(Y_C - Y_2) + (Z_2 - H_B)(Z_2 - H_C) \quad (64b)$$

and

$$\rho'_C \rho'_A \cos \beta'_{ca} = (X_C - X_2)(X_A - X_2) + (Y_C - Y_2)(Y_A - Y_2) + (Z_2 - H_C)(Z_2 - H_A) \quad (64c)$$

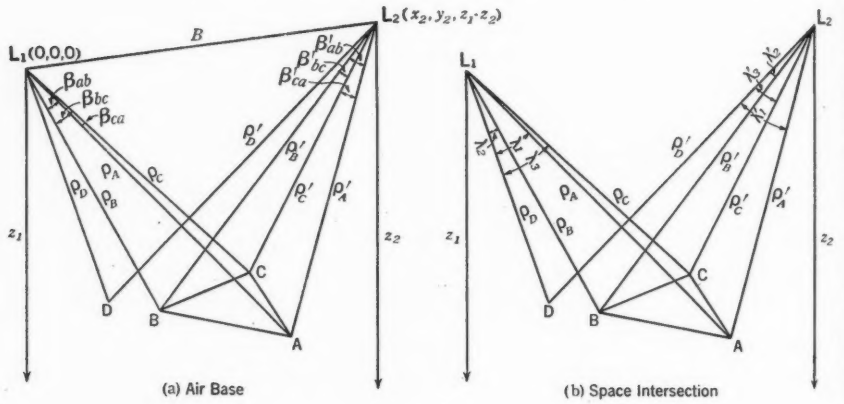


FIG. 6

Therefore,

$$\rho'_A \rho'_B \cos \beta'_{ab} - (Z_2 - H_A)(Z_2 - H_B) = (X_A - X_2)(X_B - X_2) + (Y_A - Y_2)(Y_B - Y_2) \quad (65a)$$

$$\rho'_B \rho'_C \cos \beta'_{bc} - (Z_2 - H_B)(Z_2 - H_C) = (X_B - X_2)(X_C - X_2) + (Y_B - Y_2)(Y_C - Y_2) \quad (65b)$$

and

$$\rho'_C \rho'_A \cos \beta'_{ca} - (Z_2 - H_C)(Z_2 - H_A) = (X_C - X_2)(X_A - X_2) + (Y_C - Y_2)(Y_A - Y_2) \quad (65c)$$

$$6V' = \begin{vmatrix} X_2 & Y_2 & Z_1 - Z_2 & 1 \\ X_A & Y_A & Z_1 - H_A & 1 \\ X_B & Y_B & Z_1 - H_B & 1 \\ X_C & Y_C & Z_1 - H_C & 1 \end{vmatrix} = \rho'_A \rho'_B \rho'_C \begin{vmatrix} 1 & \cos \beta'_{ca} & \cos \beta'_{bc} \\ \cos \beta'_{ca} & 1 & \cos \beta'_{ab} \\ \cos \beta'_{bc} & \cos \beta'_{ab} & 1 \end{vmatrix}^{\frac{1}{2}} \quad (66)$$

in which V' is the volume of the second ground pyramid; and H_A, H_B, and H_C are the elevations of the ground points A, B, and C, respectively. Eq. 66 is a linear equation with two unknowns of X_2 and Y_2. Since X_A, Y_A, H_A, X_B, Y_B, H_B, X_C, Y_C, H_C, Z_2, ρ'_A, ρ'_B, ρ'_C, $\cos \beta'_{ab}$, $\cos \beta'_{bc}$, and $\cos \beta'_{ca}$ now are known quantities, any one of Eqs. 65 and Eq. 66 can be solved theoretically for X_2 and Y_2. In the solution of these second degree equations, the signs

of X_2 and Y_2 can be determined in accordance with the position of the image of the second exposure station on the first photograph. Then $\overline{L_1 L_2} = B$: $X_2 i + Y_2 j + (Z_1 - Z_2) k$; and therefore the length of the "air base" will be

$$B = \sqrt{X^2_2 + Y^2_2 + (Z_1 - Z_2)^2} \quad \ldots\ldots\ldots\ldots\ldots (67)$$

The horizontal length of (B) is equal to $\sqrt{X^2_2 + Y^2_2}$; and the direction cosines of B are $\cos \mu_x = \frac{X_2}{B}$, $\cos \mu_y = \frac{Y_2}{B}$, and $\cos \mu_z = \frac{Z_1 - Z_2}{B}$.

Space Intersection.—If there is any new point (D, Fig. 6) common to the two photographs, the coordinates of its image on the first photograph with respect to the first plumb-line coordinate system will be designated as x_d, y_d, z_d; and those of its image on the second photograph with respect to the second plumb-line coordinate system, by x'_d, y'_d, z'_d. The coordinates x_a, y_a, z_a; x_b, y_b, z_b; and x_c, y_c, z_c of the images of the three ground control points A, B, and C in the first photograph, with respect to the first plumb-line coordinate system, are known. Likewise, the coordinates x'_a, y'_a, z'_a; x'_b, y'_b, z'_b; and x'_c, y'_c, z'_c in the second photograph, with respect to the second plumb-line coordinate system, are known. Taking scalar products of the vector ρ for the first set,

$$\rho_a \rho_d \cos \lambda_a = x_a x_d + y_a y_d + z_a z_d \quad \ldots\ldots\ldots\ldots\ldots (68a)$$

$$\rho_b \rho_d \cos \lambda_b = x_b x_d + y_b y_d + z_b z_d \quad \ldots\ldots\ldots\ldots\ldots (68b)$$

and

$$\rho_c \rho_d \cos \lambda_c = x_c x_d + y_c y_d + z_c z_d \quad \ldots\ldots\ldots\ldots\ldots (68c)$$

and, for the second set,

$$\rho'_a \rho'_d \cos \lambda'_a = x'_a x'_d + y'_a y'_d + z'_a z'_d \quad \ldots\ldots\ldots\ldots\ldots (69a)$$

$$\rho'_b \rho'_d \cos \lambda'_b = x'_b x'_d + y'_b y'_d + z'_b z'_d \quad \ldots\ldots\ldots\ldots\ldots (69b)$$

and

$$\rho'_c \rho'_d \cos \lambda'_c = x'_c x'_d + y'_c y'_d + z'_c z'_d \quad \ldots\ldots\ldots\ldots\ldots (69c)$$

In Eqs. 68 and 69, λ_a, λ_b, and λ_c are the apex angles between ρ_d and ρ_a, ρ_b, and ρ_c, respectively; and λ'_a, λ'_b, λ'_c, the apex angles between ρ'_d and ρ'_a, ρ'_b, and ρ'_c, respectively. Solving for x_d, y_d, and z_d in Eqs. 68 yields the coordinates of the image of point D in the first photograph and solving for x'_d, y'_d, and z'_d from Eqs. 69 yields the coordinates of the image of point D in the second photograph. Again, taking scalar products of the rays from the first exposure station L_1 to the ground points,

$$\rho_A \rho_D \cos \lambda_a = X_A X_D + Y_A Y_D + (Z_1 - H_A)(Z_1 - H_D) \quad \ldots\ldots (70a)$$

$$\rho_B \rho_D \cos \lambda_b = X_B X_D + Y_B Y_D + (Z_1 - H_B)(Z_1 - H_D) \quad \ldots\ldots (70b)$$

and

$$\rho_C \rho_D \cos \lambda_c = X_C X_D + Y_C Y_D + (Z_1 - H_C)(Z_1 - H_D) \quad \ldots\ldots (70c)$$

and, from the second exposure station to the ground points,

$$\rho'_A \rho'_D \cos \lambda'_a = X'_A X'_D + Y'_A Y'_D + (Z_2 - H_A)(Z_2 - H_D) \quad \ldots (71a)$$

$$\rho'_B \rho'_D \cos \lambda'_b = X'_B X'_D + Y'_B Y'_D + (Z_2 - H_B)(Z_2 - H_D) \quad \ldots (71b)$$

and

$$\rho'_C \rho'_D \cos \lambda'_c = X'_C X'_D + Y'_C Y'_D + (Z_2 - H_C)(Z_2 - H_D) \ldots (71c)$$

By plane trigonometry, from the first set of the relations, as shown in Fig. 6(*b*),

$$\overline{AD}^2 = \rho^2_A + \rho^2_D - 2 \rho_A \rho_D \cos \lambda_a \ldots (72a)$$

$$\overline{BD}^2 = \rho^2_B + \rho^2_D - 2 \rho_B \rho_D \cos \lambda_b \ldots (72b)$$

and

$$\overline{CD}^2 = \rho^2_C + \rho^2_D - 2 \rho_C \rho_D \cos \lambda_c \ldots (72c)$$

and, from the second set,

$$\overline{AD}^2 = (\rho'_A)^2 + (\rho'_D)^2 - 2 \rho'_A \rho'_D \cos \lambda'_a \ldots (73a)$$

$$\overline{BD}^2 = (\rho'_B)^2 + (\rho'_D)^2 - 2 \rho'_B \rho'_D \cos \lambda'_b \ldots (73b)$$

and

$$\overline{CD}^2 = (\rho'_C)^2 + (\rho'_D)^2 - 2 \rho'_C \rho'_D \cos \lambda'_c \ldots (73c)$$

After eliminating $\overline{AD}$, $\overline{BD}$, and $\overline{CD}$ from Eqs. 72 and 73,

$$\rho^2_A + \rho^2_D - 2 \rho_A \rho_D \cos \lambda_a = (\rho'_A)^2 + (\rho'_D)^2 - 2 \rho'_A \rho'_D \cos \lambda'_a \ldots (74a)$$

$$\rho^2_B + \rho^2_D - 2 \rho_B \rho_D \cos \lambda_b = (\rho'_B)^2 + (\rho'_D)^2 - 2 \rho'_B \rho'_D \cos \lambda'_b \ldots (74b)$$

and

$$\rho^2_C + \rho^2_D - 2 \rho_C \rho_D \cos \lambda_c = (\rho'_C)^2 + (\rho'_D)^2 - 2 \rho'_C \rho'_D \cos \lambda'_c \ldots (74c)$$

Eliminating ρ^2_D and $(\rho'_D)^2$ from Eqs. 74,

$$\rho^2_A - \rho^2_B + 2 \rho_D (\rho_B \cos \lambda_b - \rho_A \cos \lambda_a) = (\rho'_A)^2 - (\rho'_B)^2 + 2 \rho'_D (\rho'_B \cos \lambda'_b - \rho'_A \cos \lambda'_a) \ldots (75a)$$

and

$$\rho^2_A - \rho^2_C + 2 \rho_D (\rho_C \cos \lambda_c - \rho_A \cos \lambda_a) = (\rho'_A)^2 - (\rho'_C)^2 + 2 \rho'_D (\rho'_C \cos \lambda'_c - \rho'_A \cos \lambda'_a) \ldots (75b)$$

Let

$$\tfrac{1}{2} \{\rho^2_A - \rho^2_B - [(\rho'_A)^2 - (\rho'_B)^2]\} = w_1 \ldots (76a)$$

and

$$\tfrac{1}{2} \{\rho^2_A - \rho^2_C - [(\rho'_A)^2 - (\rho'_C)^2]\} = w_2 \ldots (76b)$$

Also, let

$$\rho_B \cos \lambda_b - \rho_A \cos \lambda_a = a_1 \ldots (77a)$$

$$\rho'_B \cos \lambda'_b - \rho'_A \cos \lambda'_a = b_1 \ldots (77b)$$

$$\rho_C \cos \lambda_c - \rho_A \cos \lambda_a = a_2 \ldots (77c)$$

and

$$\rho'_C \cos \lambda'_c - \rho'_A \cos \lambda'_a = b_2 \ldots (77d)$$

Then

$$a_1 \rho_D - b_1 \rho'_D + w_1 = 0 \ldots (78a)$$

and

$$a_2 \rho_D - b_2 \rho'_D + w_2 = 0 \ldots (78b)$$

The solution of Eqs. 78 yields the values of ρ_D and ρ'_D, after which the elevation

of points D,H_D, will be given by either of the two equations (compare Eqs. 31):

$$\rho_D = \frac{Z_1 - H_D}{z_d} \rho_d \quad \ldots\ldots (79a)$$

and

$$\rho'_D = \frac{Z_2 - H_D}{z'_d} \rho'_d \quad \ldots\ldots (79b)$$

Substituting the values of ρ_D, ρ'_D, and H_D from Eqs. 79 back into Eqs. 70 and 71, the solution of any two of Eqs. 70 gives the coordinates of point D with respect to the first plumb-line coordinate system; and the solution of any two of Eqs. 71 gives the coordinates of point D with respect to the second plumb-line coordinate system. The determination of the same point with respect to two different coordinate systems is equivalent to a determination of the angular relationship between the two systems, since their origins are known, one being referred to the other. By the standard formulas of translation and rotation of axes,

$$x_1 = x_o + x_2 \cos \alpha - y_2 \sin \alpha \quad \ldots\ldots (80a)$$

and

$$y_1 = y_o + x_2 \sin \alpha + y_2 \cos \alpha \quad \ldots\ldots (80b)$$

in which x_o,y_o are the coordinates of the origin of the second system on the first system; x_1,y_1 and x_2,y_2 are the coordinates of a specific point under consideration with reference to the two systems, respectively; and α is the angle between the corresponding axes of the two systems. Since x_o,y_o; x_1,y_1; and x_2,y_2 are known, the values of cos α and sin α can be obtained from Eqs. 80, in which x_o,y_o; x_1,y_1; and x_2,y_2 are equivalent to X_2,Y_2; X_D,Y_D; and X'_D,Y'_D, respectively, in Eqs. 63 to 71.

The procedures explained in this section can be applied to any other points that have common images on the two successive photographs.

Again, if there is a new point D, common to the two successive photographs, and this new point has its image coordinates in the first photograph, x_d,y_d,z_d, with respect to the first plumb-line coordinate system, the direction cosines of the line L_1D, from the first exposure station to the ground point D, will then be

$$\cos \alpha_1 = \frac{x_d}{\overline{L_1 d}} \quad \ldots\ldots (81a)$$

$$\cos \beta_1 = \frac{y_d}{\overline{L_1 d}} \quad \ldots\ldots (81b)$$

and

$$\cos \gamma_1 = \frac{z_d}{\overline{L_1 d}} \quad \ldots\ldots (81c)$$

in which

$$\overline{L_1 d} = \rho_d = \sqrt{x^2_{pd} + y^2_{pd} + f^2} \quad \ldots\ldots (82)$$

All the coordinates of the ground points A, B, and C and the second exposure station with respect to the first photograph plumb-line coordinate system are $X_A,Y_A,(Z_1 - H_A)$; $X_B,Y_B,(Z_1 - H_B)$; $X_C,Y_C,(Z_1 - H_C)$; and X_2,Y_2,Z_2, re-

spectively. Consequently, the formulas—

$$\frac{\rho'_a}{\rho'_A} = \frac{x'_a}{X_A - X_2} = \frac{y'_a}{Y_A - Y_2} = \frac{z'_a}{Z_1 - Z^*_2 - H_A} \quad \ldots\ldots (83a)$$

$$\frac{\rho'_b}{\rho'_B} = \frac{x'_b}{X_B - X_2} = \frac{y'_b}{Y_B - Y_2} = \frac{z'_b}{Z_1 - Z^*_2 - H_B} \quad \ldots\ldots (83b)$$

and

$$\frac{\rho'_c}{\rho'_C} = \frac{x'_c}{X_C - X_2} = \frac{y'_c}{Y_C - Y_2} = \frac{z'_c}{Z_1 - Z^*_2 - H_C} \quad \ldots\ldots (83c)$$

—give the coordinates of images a′, b′, and c′ in the second photograph with respect to a plumb-line coordinate system on which the second exposure station is taken as the origin. In Eqs. 83, Z^*_2 is not the elevation of the second exposure station, as usually designated in the preceding sections. It is really equal to $Z_1 - Z_2$ in the sense of the absolute flight heights. The three rectangular axes are parallel to the axes of the first plumb-line coordinate system. Thus, the vectors of ρ'_a, ρ'_b, ρ'_c, and ρ'_d will be as follows:

$$\rho'_a: \quad x'_a\, i + y'_a\, j + z'_a\, k \quad \ldots\ldots (84a)$$

$$\rho'_b: \quad x'_b\, i + y'_b\, j + z'_b\, k \quad \ldots\ldots (84b)$$

$$\rho'_c: \quad x'_c\, i + y'_c\, j + z'_c\, k \quad \ldots\ldots (84c)$$

and

$$\rho'_d: \quad x'_d\, i + y'_d\, j + z'_d\, k \quad \ldots\ldots (84d)$$

Taking scalar products of the vectors in Eqs. 84,

$$\rho'_a \rho'_d \cos \lambda'_a = x'_a x'_d + y'_a y'_d + z'_a z'_d \quad \ldots\ldots (85a)$$

$$\rho'_b \rho'_d \cos \lambda'_b = x'_b x'_d + y'_b y'_d + z'_b z'_d \quad \ldots\ldots (85b)$$

and

$$\rho'_c \rho'_d \cos \lambda'_c = x'_c x'_d + y'_c y'_d + z'_c z'_d \quad \ldots\ldots (85c)$$

in which λ'_a, λ'_b, and λ'_c, the corresponding apex angles with respect to the second photograph, are the known quantities. The coordinates of image d in the second photograph are computed by these three equations for x'_d, y'_d, and z'_d. Accordingly, the following equations give the direction cosines of line L_2D, Fig. 6, a line drawn from the second exposure station to the ground point D:

$$\cos \alpha_2 = \frac{x'_d}{\rho'_d} \quad \ldots\ldots (86a)$$

$$\cos \beta_2 = \frac{y'_d}{\rho'_d} \quad \ldots\ldots (86b)$$

and

$$\cos \gamma_2 = \frac{z'_d}{\rho'_d} \quad \ldots\ldots (86c)$$

By analytical geometry,

$$\frac{X_D}{\cos \alpha_1} = \frac{Y_D}{\cos \beta_1} = \frac{Z_1 - H_D}{\cos \gamma_1} \quad \ldots\ldots (87a)$$

and

$$\frac{X_D - X_2}{\cos \alpha_2} = \frac{Y_D - Y_2}{\cos \beta_2} = \frac{Z_1 - H_D - Z^*_2}{\cos \gamma_2} \quad \ldots\ldots\ldots\ldots (87b)$$

Eqs. 87 give the coordinates of point D, with respect to the first plumb-line coordinate system. Since all the coordinates discussed are on the same coordinate system and are then known, it is a simple matter to calculate the geodetic position of one point with respect to the other.

Section 7. Conclusion

The comparison of the results obtained in this paper with the results obtained by Earl Church, Assoc. M. ASCE, from his space-geometry method[8] lends confidence to a conviction that the procedure adopted in this paper as demonstrated in Section 5 is absolutely correct. For example,

Method	Z, in feet	Tilt	Swing
Church.....	20,201	0° 59.7′	180° 29.0′
Chow.....	20,198	0° 59.8′ 0° 59.6′ 1° 02.5′	179° 39.0′

If only the altitude of the exposure station and the tilt of photograph are to be found, the following simplified procedure is recommended:

(1) Measure the image coordinates with respect to the photographic axes;

(2) From the coordinates of the images find ρ_a, ρ_b, and ρ_c, l_{ab}, l_{bc}, and l_{ca}, and cos β_{ab}, cos β_{bc}, and cos β_{ca};

(3) By the successive approximation method find ρ_A, ρ_B, and ρ_C;

(4) From ρ_A, ρ_B, ρ_C, and cos τ_B (or cos τ_C or cos τ_A) calculate the altitude of the exposure station;

(5) Using the altitude of the exposure station compute cos V_a, cos V_b, and cos V_c, and then find ρ_v, x_{pv}, and y_{pv} from Eqs. 60; and

(6) Compute $\overline{Ov} = \sqrt{x^2_{pv} + y^2_{pv}}$, from which tan t is computed by Eq. 54.

This procedure involves only simple computations and affords very accurate results.

Acknowledgments.—Several methods of solving the problem outlined in this paper have appeared in the technical literature, some of which have already been cited herein; as, for example, the "space-geometry method"[2] and the "direction cosine solution"[4] devised by Professor Church. Solutions have also been offered by R. O. Anderson,[5] Assoc. M. ASCE, and P. H. Underwood,[6] M. ASCE. The writer is particularly indebted to Professor Church for his inspiration and instruction while this paper was in the course of preparation, and especially for supplying the data used in Section 5.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

PAPERS

WIND STRESSES IN SEMI-RIGID CONNECTIONS OF STEEL FRAMEWORK

BY BASIL SOUROCHNIKOFF[1]

SYNOPSIS

Much has been written on wind stresses in general and also on stresses in semi-rigid framework. Some disagreement remains, however, on how to combine the stresses due to gravity loads with those due to wind in the connections of semi-rigid framework, particularly when the connections are stressed beyond the yield point in bending. As the semi-rigid connections are often allowed to work beyond the yield point, this question is of importance. This paper presents a theoretical discussion of stresses under these conditions, and derives general conclusions applicable to rectangular framework.

INTRODUCTION

Connections of windbracing beams in steel framework are often encountered in practice, as such beams are used when X-bracing cannot be used because of clearances, when the stiffening effect of walls or partitions is considered to be insufficient, and when, at the same time, the horizontal forces are of sufficient magnitude to warrant taking special provisions to resist their action. Where wind connections are used in the bents of simple framework the girders are usually connected to the columns by top and bottom angles or tees so as to resist within certain limits the change of angle between the girders and the columns. Therefore, the connections are of the semi-rigid type and the study of stresses in them necessitates an analysis of semi-rigid framework.

Study of the stresses in semi-rigid framework has been receiving attention for many years, and many methods of analysis, both exact and approximate, have been developed. However, there are few data on the manner of combining wind stresses with gravity stresses, and there is considerable disagreement among designers on this point.

In its 1946 code[2] the American Institute of Steel Construction (AISC) recommends that wind connections be adequate to resist moments induced by

NOTE.—Written comments are invited for immediate publication; the last discussion should be submitted by July 1, 1949.

[1] Structural Designer, Eng. Dept., E. I. du Pont de Nemours & Co., Wilmington, Del.

[2] "Code of Standard Practice for Steel Buildings and Bridges," A.I.S.C., 1946.

gravity loads and wind loads at the increased wind stresses permitted therefor, or that the connections carrying wind moments be so designed that larger moments induced by the gravity loads under the actual conditions of restraint will be relieved by deformation of the connecting material without overstress of the welds. Many designers, however, use an approximate method in which the stresses due to gravity loads are entirely neglected and the connections are designed for wind moments alone. Other designers take the opposing view and recommend that the connections be designed for the algebraical sum of stresses caused by gravity loads and wind loads computed independently of each other with the assumption in each case that the connections act elastically.

Although the last two views seem to contradict each other, each one may hold true for a certain degree of flexibility of the connections. When the connections are very flexible—that is, if they can undergo large inelastic deformations without failure—it may be permissible to neglect the effect of gravity loads. On the other hand, if the breaking point is reached at a comparatively small deformation beyond the elastic limit, both the gravity loads and the wind loads must be considered.

The purpose of this paper is to analyze the manner in which gravity stresses are combined with wind stresses after the elastic limit has been reached.

Summary of Data on the Design of Semi-Rigid Framework

Available data on the design of semi-rigid frames can be divided into two sections for examination: (*a*) The properties of semi-rigid connections, and (*b*) the analysis and design of semi-rigid framework.

Properties of Semi-Rigid Connections.—The deformation of a semi-rigid connection is too complex to be determined mathematically. In fact, it depends on the deflection of the top and bottom angles or tees, the deflection of the column flange to which it may be connected, and the deformation and possible slip of the shear rivets. The last factor cannot be represented by a simple mathematical expression. Therefore, the experimental determination of stress-strain curves of the type given by J. Charles Rathbun,[3] M. ASCE, and J. L. Brandes, Jun. ASCE, and R. M. Mains,[4] Assoc. M. ASCE, is of value.

The stress-strain curves are plotted with the moment M acting on the connection as ordinate and the rotation α of the connection as abscissa. These curves have a generally parabolic shape (Fig. 1), the rotation ceasing to be proportional to the moment at rather low values of the moment, so that for most of the length of the curve the slope, $1/Z$, is not constant. However, up to a certain value of the moment the change in Z is small and the connection within these limits may be considered to be elastic. As a rule, the curves do not show a definite yield point comparable to the one that appears on the stress-strain curves of steel tested in tension, and the variation of Z is progressive. When the connection is loaded to a point such as C (Fig. 1) and then unloaded, the representative point travels along a line CC′, which is practically straight. Some irregularities occur in this line, however, especially in the

[3] "Elastic Properties of Riveted Connections," by J. Charles Rathbun, *Transactions*, ASCE, Vol. 101, 1936, p. 524.

[4] "Report of Tests of Welded Top-Plate and Seat Building Connections," by J. L. Brandes and R. M. Mains, *Welding Journal*, March, 1944, p. 146-s.

vicinity of point C′, where the moment is zero, and are probably due to the slip that occurs in the rivets when the load is reversed. All unloading lines of a connection are closely parallel to each other, and the value of Z of the unloading lines depends but little on the point C reached during loading. Apparently, not many tests have been made on the behavior of a connection under the reversal of moment, or on the fatigue strength of a connection. Professor Rathbun[3] gives stress-strain curves of two specimens under reversal of moment. These curves show that after the connection is unloaded along a line such as CC′ (Fig. 1) and the moment is reversed, the stress-strain curve under negative moment is somewhat similar to the original loading curve; that is, the slope of the curve decreases gradually with the increase of the negative moment. After reaching a point such as C″ on this curve, if the connection is again unloaded—that is, if the negative moment is reduced—the new unloading curve C″ C‴ is practically a straight line and is closely parallel to the unloading line CC′. If the load is reversed, the corresponding curve C‴ C⁗ is closely similar to loading curve OC.

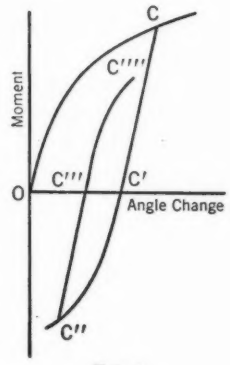


FIG. 1

Analysis and Design of Semi-Rigid Framework.—The various methods of analysis and design may be classified into two types depending on whether or not the rotation of the connections is assumed to be proportional to the moment.

In the first case, in which the rotation of the connections is assumed to be proportional to the moment, the structure as a whole, including connections, is considered to be elastic. A connection is thought of as a locally weakened section between the end of the beam and the face of the column, the effect of which is the inverse of the effect produced by end haunches or added cover plates. The structure as a whole is therefore a rigid frame comprising a number of locally weakened sections, and the ordinary methods of analysis of rigid frames, such as the slope-deflection method or the moment-distribution method, are applicable. However, the stiffness and distribution factors must be adjusted to provide for the local weakening of sections at connections.[3,5,6,7]

In the second case, since the stress-strain curves have no simple mathematical expression, the structure must be analyzed either by successive approximations or by graphical methods, as indicated in a British report.[8]

Analysis of Gravity and Wind Stresses in Connections Working Beyond the Proportional Limit

The types of design methods previously mentioned are rather complicated and time consuming in their application, and they assume that the properties

[5] "Analysis of Building Frames with Semi-Rigid Connections," by Bruce Johnston and Edward H. Mount, *Transactions*, ASCE, Vol. 107, 1942, p. 993.
[6] "The Analysis of Rigid Frames," by A. Amirikian, U. S. Govt. Printing Office, Washington, D. C., 1942, p. 338.
[7] "Theory of Modern Steel Structures," by L. E. Grinter, The Macmillan Co., New York, N. Y., Vol. 2, 1936.
[8] "Second and Final Reports," Steel Structures Research Committee, Dept. of Scientific and Industrial Research of Great Britain, H. M. Stationery Office, London, England, 1934.

of the connections are well known. These properties, however, are known only in the case of a few typical connections, and even for these connections they depend on the history of the connection—that is, on the previous loading the connection has sustained and on the permanent set it may have taken. Therefore, too much refinement in the analysis is not warranted.

It is claimed[4,5,9] that, as a result of partial continuity, a saving of from 15% to 20% of steel can be made by using semi-rigid connections in place of standard connections. In practice, however, the semi-rigid connections are used mostly to resist side loads only, the possible saving of steel being of secondary importance. It seems, therefore, that it would be useful to have a design method that would permit checking a connection for overstress under the combination of gravity and side loads without going through the process of exact determination of the stresses caused by gravity loads. The usual design method, which neglects entirely the stresses from gravity loads, is of this type. Its drawback, however, is that it may be open to question as to just why the frame should be considered to be rigid for side loads and, at the same time, pin connected for gravity loads.

In the following analysis, in which an attempt is made to determine the limits within which the usual design method is justifiable, it is assumed that:

1. The stress-strain diagram is of the general type shown in Fig. 1, although the exact shape and the numerical relationship between the angle of rotation and the corresponding moment may not be known for all points of the curve.
2. The connection can undergo a considerable deformation without failure. The connections described by Professor Rathbun[3] and by Mr. Brandes and Professor Mains[4] are of this type. However, if the connection is made more rigid by welding, by using splice plates instead of top angles, or by other methods, the frame will be classified as a rigid frame rather than a semi-rigid one, and the analysis will not apply.

It is assumed first that the moments in the connections due to the combination of gravity and side loads are small enough so that the rotations can be considered to be proportional to the moments. In this case, the structure as a whole is elastic and the moments can be obtained by one of the standard methods of analysis of rigid frames, taking into account the local reduction of the rigidity of the beams at the connections. Under these conditions, obviously, the stress from the side loads may be computed independently and added algebraically to the stress caused by the gravity loads to obtain the total stress. If the total stress exceeds the proportional limit, the rigid frame method is not applicable; but the frame is not necessarily unsafe, as a certain amount of nonelastic deformation is acceptable in semi-rigid frames. In order to visualize what happens in semi-rigid frames in which the connections are stressed beyond the proportional limit, two examples are analyzed: First, a preliminary example of a symmetrical rectangular frame under symmetrical gravity loads and a side load; and, second, a general case of a semi-rigid frame under gravity loads and side loads. It is assumed in both cases that the connections are stressed above

[9] "Design Economy by Connection Restraint," by Bruce Johnston and Robert A. Hechtman, *Engineering News-Record*, October 10, 1940, p. 74.

the proportional limit under gravity loads alone, and that the side load enhances this condition. The same analysis holds, however, if the connections are stressed below the proportional limit under gravity loads and the proportional limit is exceeded only when the side load is applied. In both cases, it is assumed also that the side load may act in either direction; that is, it may be reversed after being applied in a given direction, then removed and reapplied in either direction.

Symmetrical Rectangular Frame Under Symmetrical Vertical Load and a Side Load.—In a rectangular bent ABB′A′ under symmetrical vertical load (Fig. 2(*a*)), because of the partial restraint at the connections A and B there will be

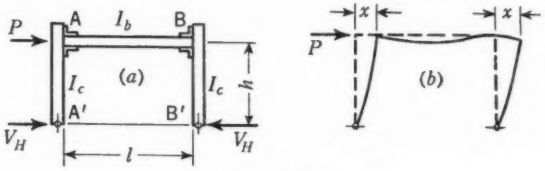


FIG. 2

a restraining moment M_0 in these connections. Therefore, there will also be equal and opposite shearing forces V_H in the columns. The state of the connections may be represented by the point C_0 on the stress-strain diagram (Fig.

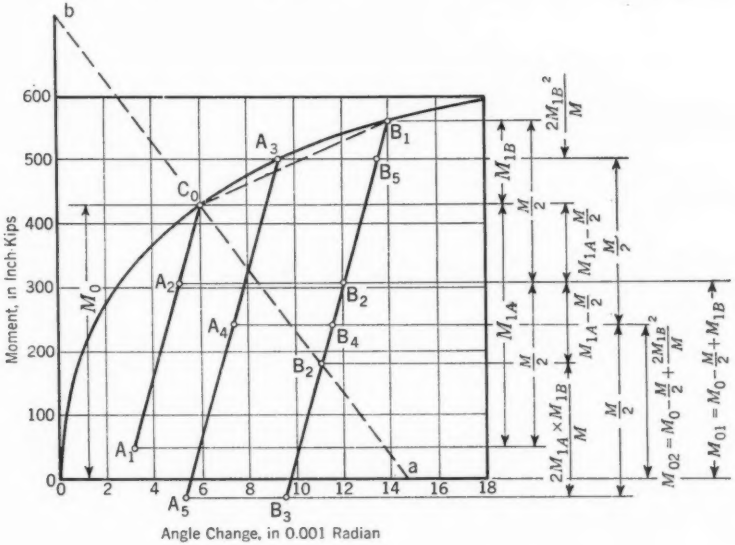


FIG. 3
(*Correction:* The lower point B_2 should be B'_2)

3). This point, and M_0, can be determined either by successive approximations or graphically as described elsewhere.[8]

It is now assumed that a side load P is applied at point A. The angle at point A will open by a quantity α_1, and the moment will decrease by M_{1A} and

become $M_0 - M_{1A}$. The point representing the connection at A will move to A_1 on the return branch of the stress-strain curve. As this line is practically straight,

$$\alpha_1 = Z\, M_{1A} \quad \ldots\ldots (1a)$$

Similarly, the angle at point B will close by a quantity α_2, and the moment will increase by M_{1B} and become $M_0 + M_{1B}$. The point representing the connection at B will move to B_1 on the original stress-strain curve. It may be assumed that the section C_0B_1 of the curve is practically straight, so that

$$\alpha_2 = Z_1\, M_{1B} \quad \ldots\ldots (1b)$$

The angle changes α_1 and α_2 also can be determined from the deflection of the structure due to moments M_{1A} and M_{1B} by ordinary formulas of the strength of materials. Thus,

$$\alpha_1 = \frac{x}{h} - \left(\frac{l}{3\,E\,I_b} + \frac{h}{3\,E\,I_c}\right) M_{1A} + \frac{l}{6\,E\,I_b}\, M_{1B} \quad \ldots\ldots (2a)$$

and

$$\alpha_2 = \frac{x}{h} - \left(\frac{l}{3\,E\,I_b} + \frac{h}{3\,E\,I_c}\right) M_{1B} + \frac{l}{6\,E\,I_b}\, M_{1A} \quad \ldots\ldots (2b)$$

Eliminating α_1 and α_2 from Eqs. 1 and 2,

$$\frac{M_{1A}}{M_{1B}} = \frac{\dfrac{h}{3\,E\,I_c} + \dfrac{l}{2\,E\,I_b} + Z_1}{\dfrac{h}{3\,E\,I_c} + \dfrac{l}{2\,E\,I_b} + Z} \quad \ldots\ldots (3)$$

Because of the side loads the shear at point A′ will decrease by M_{1A}/h and the shear at point B′ will increase by M_{1B}/h. The total variation is equal to the side load P, so that

$$P = \frac{M_{1A} + M_{1B}}{h} \quad \ldots\ldots (4)$$

From Eqs. 3 and 4 it is possible to derive

$$M_{1A} = \frac{P\,h}{1 + M_{1B}/M_{1A}} \quad \ldots\ldots (5a)$$

and

$$M_{1B} = \frac{P\,h}{1 + M_{1A}/M_{1B}} \quad \ldots\ldots (5b)$$

Inasmuch as $Z_1 > Z$, it follows that $M_{1A}/M_{1B} > 1$ and $M_{1B} < 1\ M_{1A}$. This means that the side load mostly decreases the moment at point A whereas the increase of moment at point B is comparatively small. It should be noted that if the connections at point A and point B were rigid—that is, in the case of a rigid frame—the decrease of the moment at A would be equal to the increase of moment at B and would have the value $P\,h/2$. This shows that the increase of moment in a connection caused by the side load is smaller in a semi-rigid frame than in a rigid frame of the same dimensions and subject to the same load.

It is next assumed that the side load is removed. The frame will sway back, but will not reach the original position because it has undergone a plastic deformation when the side load was first applied. The point representing the connection at point B will travel from B_1 to B_2 on the return branch of the stress-strain curve (Fig. 3); the point representing the connection at point A will travel from A_1 to A_2. Both connections act elastically. The total variation of moment is

$$M = P\,h \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

and is divided equally between the two connections. It is seen that the moments restraining the beam at A and B become smaller than they were before application and removal of the side load, their value becoming

$$M_{01} = M_0 - M/2 + M_{1B} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

This means that if restraining moments are depended on to reduce the stress in the middle of the beam, as is done in the rigid frames, it should be remembered that the restraining moments are reduced, and, consequently, the factor of safety at the midspan of the beam is also reduced, because of the application and subsequent removal of a side load. If the side load is again applied in the same direction, the connections will again act elastically, and the points reached on the stress-strain curve will again be A_1 and B_1.

On the contrary, if from the unloaded position the side load is applied in the direction opposite to the first load—that is, in the direction BA—the moment in the connection at point A will increase and the moment in the connection at point B will decrease. The point representing the connection at A will move from A_2 to C_0 along the elastic line, then to A_3 on the plastic line. The point representing the connection at B will move from B_2 to B_3 on the elastic line, reaching point B'_2 when point C_0 is reached by the point representing the connection at A. When points C_0 and B'_2 are reached, the total change of moment in both connections is $2\,(M_{1A} - M/2)$ and there remains a moment of

$$M - 2\,M_{1A} + M = 2\,(M - M_{1A}) = 2\,M_{1B} \ldots\ldots\ldots\ldots (8)$$

which will be taken up by the movement of the representative points from C_0 to A_3 and from B'_2 to B_3, respectively. As the connection at point A acts plastically from point C_0 on, this remaining moment will be distributed between the connections at point A and at point B in the ratio M_{1B}/M_{1A}, so that the increment M_{2A} of moment in the connection at point A for the movement from C_0 to A_3, and the decrement M_{2B} of the moment in the connection at point B for the movement from B'_2 to B_3 will be determined by

$$M_{2A} + M_{2B} = 2\,M_{1B} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (9a)$$

and

$$M_{2A}/M_{2B} = M_{1B}/M_{1A} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (9b)$$

Thus,

$$M_{2A} = \frac{2\,M_{1B}}{1 + M_{1A}/M_{1B}} = \frac{2\,M^2_{1B}}{M} \ldots\ldots\ldots\ldots\ldots (10a)$$

and

$$M_{2B} = \frac{2\,M_{1B}}{1 + M_{1B}/M_{1A}} = \frac{2\,M_{1A}\,M_{1B}}{M} \quad \ldots\ldots\ldots\ldots (10b)$$

Eqs. 10 determine the location of points A_3 and B_3, and therefore the moments in the connections at this time.

If the side load is now removed, the frame will sway back in the direction AB, the moment in the connection at point A will decrease, and the representative point will move from A_3 to A_4. Also, the moment in the connection at B will increase and the representative point will move from B_3 to B_4. Both connections behave elastically, so that the variation of moment is $M/2$ in each connection. It will be seen that the moment restraining the beam at points A and B is decreased still further and becomes

$$M_{02} = M_0 + \frac{2\,M^2{}_{1B}}{M} - \frac{M}{2} \quad \ldots\ldots\ldots\ldots\ldots (11)$$

From points A_4 and B_4 the side load can again be applied in either direction AB or direction BA. Both connections will work elastically, the representative points moving from A_4 to A_3 and from B_4 to B_3 for the load in the direction BA, or from A_4 to A_5 and from B_4 to B_5 for the load in the direction AB. No additional plastic deformation will occur during these and subsequent loadings.

From the preceding it may be deduced that:

a. The maximum increase of moment in the connections due to side load is M_{1B} determined by Eqs. 3 and 5 and is smaller than the increase of $M/2$ which would occur if the frame were perfectly elastic and if the moment due to side load were algebraically added to the moment due to gravity load considered independently.

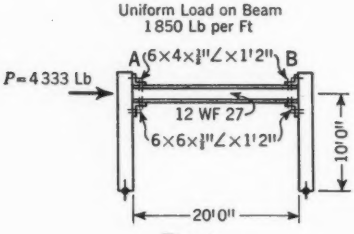


FIG. 4

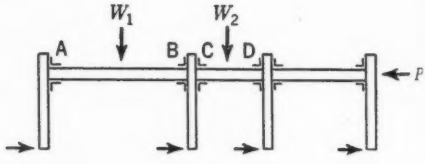


FIG. 5

b. The moments restraining the beam at A and B decrease because of the application, reversal, and removal of the side load, the value of these reduced moments being given by Eq. 11.

c. After the first application, reversal, and removal of the side load, the connections act elastically, and the increment or decrement of moment in each connection is $M/2$, counted from the mean value of moment given by Eq. 11, rather than from the original restraining moment caused by the gravity loads only.

Numerical Example.—The frame to be analyzed is shown in Fig. 4. To simplify computations, the columns are assumed to be perfectly rigid. The

stress-strain diagram of the connection is shown in Fig. 3, which is taken from Professor Rathbun's[3] specimen 10. The following computations are self-explanatory:

$$\text{Fixed-end moment} = 1{,}850 \times 20^2 \times 12/12 = 740 \text{ in-kips}$$
$$\text{Free-end rotation} = w\,l^3/24\,E\,I$$
$$= (1{,}850 \times 20^3 \times 12^2)/(24 \times 30 \times 10^6 \times 204.1)$$
$$= 0.0145 \text{ radian}$$

In Fig. 3 beam line, ab, determines the actual end moment $M_0 = 435$ in-kips. Assume that the side load is such that point B_1 is reached by the point representing the connection at B; that is, the total moment in the connection at point B is 565 in-kips. Then $Z_1 = \dfrac{0.014 - 0.006}{565 - 435} = 6.15 \times 10^{-8}$ radians per in-kip. From the return line of the stress-strain diagram $Z = 0.755 \times 10^{-8}$ radians per in-kip, and from the properties of the beam $\dfrac{l}{2\,E\,I} = \dfrac{20 \times 12}{2 \times 30 \times 10^6 \times 204.1}$ $= 1.96 \times 10^{-8}$ radians. The ratio of wind moments is $M_B/M_A = (0.755 + 1.96)/(6.15 + 1.96) = 0.333$. For the assumed $M_B = 130$ in-kips, it is found that $M_A = 390$ in-kips and $M = 520$ in-kips. Note that if the connections were assumed to be elastic, $M_{1B} = M_{1A} = 520/2 = 260$ in-kips. The slope of the line C_0A_3 is slightly different from the slope of C_0B_1. Assume that the two slopes are equal so that Eq. 11 may be used. The final end restraint moment is then $435 - 520/2 + 2 \times 130^2/520 = 240$ in-kips instead of the original 435 in-kips, a reduction to 55% of its original value.

General Case of Semi-Rigid Framework Under Vertical Loads and Side Loads.—Under gravity loads, the connections in a semi-rigid framework as a rule will exert restraining moments on the beams. When side loads are applied, these moments are changed. In some of the connections the moments are increased because of the side loads, and in other connections the moments are reduced. For example, in the frame shown in Fig. 5 there are negative moments in the connections at A, B, C, and D due to vertical loads W_1 and W_2. When a horizontal load P is applied, the frame tends to sway in the direction of P, and therefore the moments in the connections at A and C will be increased whereas the moments in the connections at B and D are decreased. It is assumed that the curves of Figs. 6(*a*) and 6(*b*) show the stress-strain diagrams of two typical connections at A and B, and that the connection at A is typical of all the connections in which the moment is increased because of a side load acting in a given direction, whereas the connection at B is typical of all the connections in which the moment is decreased because of this load. Suppose that points A_0 (Fig. 6(*a*)) and B_0 (Fig. 6(*b*)) are reached on these diagrams under vertical load only. When the side load is applied, the moment in the connection at point A is increased, and the new point reached will be A_1, located on the original stress-strain curve. At the same time the moment in the connection at point B is decreased and the new point reached will be B_1, located on the return branch of the diagram.

Now assume that the side load is removed. The moment in the connection at point A is decreased and the new point reached on the diagram is A_2, on

the return branch from A_1. Similarly, the moment in the connection at point B is increased and the new point reached on the diagram is B_2. During unloading, the rigidity of the connection at point B is practically the same as during loading, but the rigidity of the connection at point A is greater during

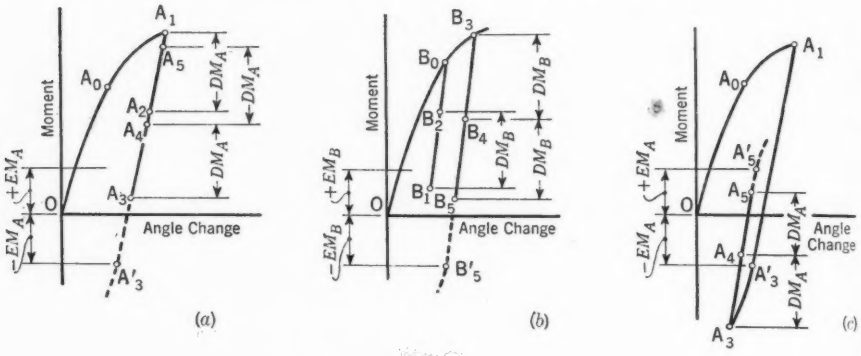


FIG. 6

unloading than during loading. Therefore, the variation of moment during unloading is not the same as during loading, and the state of the connections after the side load is removed is not the same as it was before the side load was applied. This new state is represented by points A_2 and B_2 instead of by points A_0 and B_0. It is seen, incidentally, that the restraining moment at the ends of the beams becomes smaller after the side load has been applied and subsequently removed.

By applying the side load again in the same direction as the original load, the moment in the connection at point A increases and the representative point moves from A_2 to A_1. On the other side, the moment in the connection at point B is decreased and the point moves from B_2 to B_1. The structure will then be in the same state of loading, stress, and deformation as when the side load was first applied. There is no additional plastic deformation, and the load can again be removed and applied without further plastic deformation.

Suppose now that to the horizontally unloaded structure there is applied a side load of the same intensity as the original side load, but of opposite direction. The moment in the connection at point A will decrease, and the representative point will move from A_2 to A_3. The moment in the connection at point B will increase and the point will move from B_2 toward B_0, possibly going beyond B_0 to B_3 on the plastic line. The structure may therefore undergo an additional plastic deformation. When the structure is unloaded, the representative points will move, respectively, to A_4 and B_4. During this unloading the rigidity of the connections is not the same as during loading and therefore there will remain a residual strain, the point A_4 being below the point A_2. It is seen again that the restraining moment against gravity loads is further reduced, and that, as before, the structure can again be loaded in the same directions and unloaded without any new plastic deformation, the representative points traveling, respectively, between A_4 and A_3, and between B_4 and B_3. If, how-

ever, from the last unloaded state the structure is loaded in the same direction as the original load—that is, in the direction opposite to the load just discussed —the moment in the connection at point A will increase and the representative point will move from A_4 to A_5. The moment in the connection at point B will decrease and the representative point will move from B_4 to B_5. Both connections will act elastically, and therefore the distance A_4A_5 is equal to A_2A_1, since in the movement from A_2 to A_1 and from B_2 to B_1 both connections were also acting elastically. Therefore, point A_5 is below A_1 on the elastic line.

Let $D\,M_A$ (Fig. 6(a)) and $D\,M_B$ (Fig. 6(b)) be the variation of moments in the connections at point A and point B, respectively, due to the side load, in the case when all connections of the structure are acting elastically. Also, let $E\,M_A$ (Fig. 6(a)) and $E\,M_B$ (Fig. 6(b)) be the proportional limits on the stress-strain diagram below the line of zero moments. Furthermore, let M_{A1}, M_{B1}, $\cdots$, M_{An}, and M_{Bn} be the total moments corresponding to points A_1, B_1, $\cdots$, A_n, and B_n.

Referring to Fig. 6(b), the lowermost point, B_5, will still be within the elastic range of the line B_3B_5 if B_5 is above B'_5; that is, if

$$M_{B5} > -E\,M_B \qquad (12a)$$

As

$$M_{B5} = M_{B3} - 2\,D\,M_B \qquad (12b)$$

this condition becomes

$$M_{B3} - 2\,D\,M_B > -E\,M_B \qquad (12c)$$

or

$$D\,M_B < \tfrac{1}{2}\,(M_{B3} + E\,M_B) \qquad (12d)$$

Point B_3 was assumed to be beyond the proportional limit; that is,

$$M_{B3} > E\,M_B \qquad (13a)$$

therefore, point B_5 will be within the elastic range of the line B_3B_5 if

$$D\,M_B < \tfrac{1}{2}\,(E\,M_B + E\,M_B) \qquad (13b)$$

or

$$D\,M_B < E\,M_B \qquad (13c)$$

In Fig. 6(a), the lowermost point, A_3, will still be within the elastic range of the line A_1A_3 if A_3 is above A'_3; that is, if

$$M_{A3} > -E\,M_A \qquad (14a)$$

or

$$D\,M_A < \tfrac{1}{2}\,(M_{A5} + E\,M_A) \qquad (14b)$$

It was assumed that M_{A1} is greater than $E\,M_A$, but it is not known if M_{A5} is greater than $E\,M_A$ or not. Therefore, the conclusion arrived at for the connection at point B does not necessarily hold for the connection at point A.

Assume now that A_3 is outside the elastic range, as shown in Fig. 6(c). Then A_5 will be within the elastic range of the line A_3A_5 if

$$M_{A3} + 2\,D\,M_A < E\,M_A \qquad (15a)$$

As M_{A3} is assumed to be smaller than $-E M_A$, the condition holds if

$$-E M_A + 2 D M_A < E M_A \quad \ldots\ldots (15b)$$

that is,

$$2 D M_A < 2 E M_A \quad \ldots\ldots (15c)$$

or

$$D M_A < E M_A \quad \ldots\ldots (15d)$$

Therefore, after the first horizontal load is applied, reversed, and removed, the connections will work within the elastic range for all subsequent applications of load of a magnitude not exceeding the original load provided that

$$D M_A < E M_A \quad \ldots\ldots (16a)$$

and

$$D M_B < E M_B \quad \ldots\ldots (16b)$$

In some cases, the connections may work within the elastic range even if these conditions are not satisfied.

In the preceding it was assumed that points A_1 and B_3 were reached without failure of the connections. This assumption is justified only in case the connections can undergo large deformations before failure. It is seen that $M_{A1} - M_{A0}$ and $M_{B3} - M_{B0}$ are smaller than $D M_A$ and $D M_B$, respectively, and that therefore the maximum increment in moment in the connection is smaller than the increment computed on the assumption that the connections are elastic. Likewise, it is seen that A_4 is below A_0 and B_4 is below B_0 and, therefore, that the wind load moments tend to reduce the restraining moment at the ends of the beams.

Conclusions

Further experimental determination of the behavior of semi-rigid connections under reversing moments will be helpful in establishing a correct design method. Assuming, however, that the connections can safely sustain repeated reversing stresses not exceeding a certain percentage of the proportional limit of the connections hardened by the initial strain, the following conclusions may be drawn:

(1) When connections are stressed beyond the proportional limit, the distribution of wind moments is not the same as it would be if the connections were elastic. The increments of moments above those due to gravity loads are smaller, whereas the decrements are larger.

(2) The moments restraining the ends of beams against gravity loads are reduced after wind loads have been applied and removed.

(3) In checking the strength of connections it is desirable to analyze moments caused by gravity loads as well as those caused by wind load. However, if the connections are such that they can undergo large inelastic deformations without failure, it will be found that, if the connections are adequate to resist wind stress moments alone at design stresses computed on the assumption that the connections are elastic, they are also adequate for the combination of gravity loads and wind loads.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

THE SIGNIFICANCE OF PORE PRESSURE IN HYDRAULIC STRUCTURES

Discussion

BY PAUL E. RAES

PAUL E. RAES.[52]—It seems obvious that the theory advocated by the author involves, not only a particular method for the calculation of concrete dams, but also an entirely new law of physics, applicable to porous solids. Therefore, it should be scrutinized severely before being accepted by the profession.

Internal Bouyancy.—To demonstrate his concept, Mr. Harza subdivides the material into small "columns" terminating in pores at both ends, and hence subjected to the difference of the hydrostatic pressure developed in those pores. This is called "internal buoyancy" and the author concludes (see heading, "Buoyancy"):

> "Likewise, since each column is buoyant, and the whole object is composed of such columns, then the entire internal structure is buoyant, and the effect of cementation is nullified."

In attempting to check the verity of this statement, one should remember that Archimedes' law has been demonstrated for a free body, completely submerged in still water, and not for a part of a solid limited by an imaginary surface, where elastic stresses must also be taken into account. Consider, for instance, the prism represented by Fig. 32(*a*), containing two spherical holes filled with a fluid under unit pressures p and p'.

The effect of the "internal forces" on the inner surface of the holes will be purely a change in the stress distribution; the equilibrium of the prism as a whole, and its resistance to sliding and to overturning, will remain unchanged. On the other hand, if column C is isolated, pressure p will indeed act downward on the horizontal section passing through the middle of the upper hole, and

NOTE.—This paper by L. F. Harza was published in the December, 1947, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: June, 1948, by William P. Creager, J. S. Kendrick, Adolf A. Meyer, E. Montford Fucik, A. C. R. Albery, and W. H. R. Nimmo; September, 1948, by John S. Cotton, Serge Leliavsky Bey, A. H. Davison, R. E. Ballester, John S. McNown, and V. T. Boughton; October, 1948, by Ross M. Riegel, and Douglas McHenry; November, 1948, by Frederick L. Hotes, R. W. Carlson and Raymond E. Davis, and W. G. Morrison; and December, 1948, by Raymond A. Hill, and John V. Spielman.

[52] Prof. of Soil Mechanics, Univ. of Ghent, Ghent, Belgium.

pressure p' will act upward on the corresponding section of the lower hole. If an equation of equilibrium is written by projection on a vertical axis, the shear stresses on the lateral surface of the column must be introduced.

Statics can determine the resultant of these stresses, but not their distribution; nevertheless it is likely that the pressures on the bases of the cylinder will

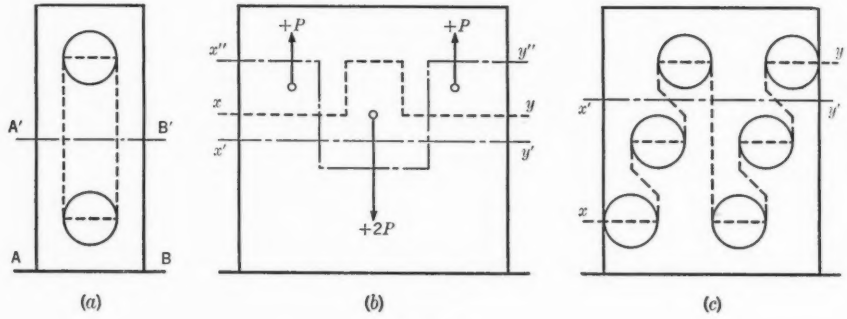


FIG. 32

be nullified by shear stresses transmitted to the parts of the lateral surface situated near both ends, in such a way that a slice at height A′ B′ (Fig.32(*a*)) will not be affected by the pore pressure. Passing to the extreme case, imagine that the fluid is enclosed in a strong casing touching the substance of the prism, but not pressing upon it, with the evident result that column C would bear no pressure at all, and could not possibly be called "buoyant." When weight is neglected, pore pressure can have no effect on the conditions existing in sections that do not intersect the pores under consideration, except through local stress variations that may occur without modifying the resulting force and couple of the stress system, in spots where pores are very near the section surface.

Consequently, the writer believes that the concept of "internal buoyancy" is not only needless, but sometimes misleading, and should be avoided, although consideration of buoyant elementary columns gives correct results if the lateral stresses nullify themselves. Indeed, under the heading, "Bottom Uplift," the author uses the notion of internal buoyancy to compute the total load on the supporting surface of a submerged block, and reaches the conclusion that there is a resultant partial uplift.

As the same calculation can be made for any other plane horizontal section cutting voids, the claim of full uplift is certainly not justified in all cases.

Uplift on a Given Surface.—In fact, full uplift is found on a very unusual surface, which has been visualized with much ingenuity by the author, and which the writer will call the "Harza surface" in this discussion. It is obtained by visualizing a block of columns terminating in pores. The lateral shear stresses disappear, of course, where two of these columns touch one another; but in the base of the block, constituting the Harza surface, certain lateral facets remain uncovered, and vertical shear stresses can still exist.

In 1943 the writer gave the name "pseudo-facets"[53] to small surface elements intersecting granular material and conforming as nearly as possible to a plane, but cutting no soil grains. The Harza surface can be considered as a generalization of those "pseudo-facets," completed by facets intersecting miscrosolids.

If different sections are considered, the forces applied on one side of the surface of separation may change, not only in the case of pore pressure, but in a general way, as will be noted in Fig. 32(*b*), where the sum of the forces is $2\,P$ above section xy, zero above section $x'y'$, and $-2\,P$ above section $x''y''$. Indeed, one can trace the pseudo-facets with the intention of "catching" certain forces and "dodging" others. If, by doing so, the points of application of these forces are not far apart, the surfaces selected nevertheless can be very close to one another.

Thus, the author has succeeded in discovering a surface subjected to full uplift; and Fig. 32(*c*), where tubular voids are assumed, shows that it is even easy to draw a surface such as section xy, submitted to 150% uplift, or more. The latter surface is composed of vertical and sloping solid facets; but this is no obstacle, as long as stresses are ignored. However, on plane $x'y'$ the uplift will be zero. The Archemedian uplift still being exactly 100%, the foregoing examples demonstrate that the term "buoyancy" cannot successfully be applied to pore pressure, and that discussion about the true value of pore pressure is meaningless as long as the exact surface of separation is unknown.

Of course, since only the vertical components of pressures are reckoned with, the horizontal components have no importance; and, if all the facets inaccessible to the liquid are vertical (which is precisely the case for the Harza surface), the total uplift will be equal to the buoyancy effort. This case is exceptional and does not justify confusion between pore pressure uplift and buoyancy.

In the paper the ultimate aim is to compute normal stresses in a horizontal section, as established by Fig. 9 and by the fact that the theory of the "middle third" is considered. At first thought, it appears to be axiomatic that, for this purpose, only the geometrical horizontal plane will suffice. Thus, uplift would act on the fraction n of the area, if n is the ordinary porosity, according to Delesse's law; but it must be remembered, as Karl Terzaghi, Hon. M. ASCE, made clear, that the so-called "strength" of a porous material is nothing more than the resistance of a surface of failure, which is not a plane, but the weakest section that can be drawn through the existing (and unknown) pore structure.

Hence, the "effective" uplift will be measured by the effective porosity n_w of the potential surface of rupture, just before it gives way, and this measurement can only be made experimentally. In this connection, the writer wishes to observe that the real surface of failure is not necessarily identical for a dry sample and for that same sample under pore pressure, because, by altering the section in the second case, the uplift may change.

Interpretation of Tests.—To determine n_w Professor Terzaghi and S. Leliavsky Bey, M. ASCE, performed clever experiments, and found, respect-

[53] "Théorie des Pseudo-Facettes en Mécanique du Sol," by P. E. Raes, *Annales des Travaux Publics de Belgique*, August, 1943.

ively, average values of 0.97 (série I), 0.97 (série II), and 0.998 (série III) in the Terzaghi compression tests,[29] and of 0.915 in the Leliavsky tension tests.[8]

From observation of Fig. 18(*c*), representing the Terzaghi surface, one understands why the experimenter assumes that the strength of the solid facets is only of a tensile nature, and is amazed by its magnitude. The structure is indeed visualized as an aggregate of grains, "welded" together in their contact points; and, in such a material, facets subjected to shear are unlikely in a section resembling AB.

On the contrary, if the material is assumed to be a continuous solid with holes in it, the separation surface may contain facets resisting through shear, and can even be constructed entirely of similar facets, like the Harza surface. Under such conditions it is evidently needless to imagine threads of astounding tensile strength when n_w approaches unity, and the value $n_w = 1$ is easily explained. However, this explanation does not demonstrate that the Harza surface is the true section of rupture.

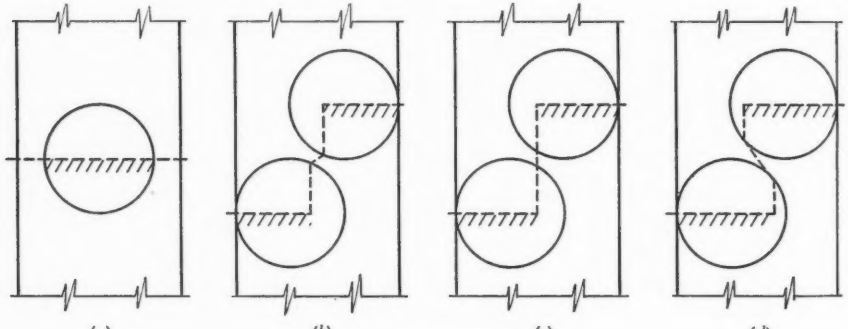


FIG. 33

In Fig. 33 different types of possible failure are suggested, and in each case the area of uplift is shaded. In Figs. 33(*a*) and 33(*b*), $n_w < 1$; in Fig. 33(*c*), $n_w = 1$; and, in Fig. 33(*d*), $n_w > 1$.

The last type, $n_w > 1$ (which is likely to occur only in compression failure), is interesting because it shows that n_w can become greater than unity, when such facets dominate. Some explanation of this kind had to be discovered, since values of 1, 1.07, and even 1.2 were found for n_w in the Terzaghi tests, and would be simply bewildering if only tensile strength was possible. The fact that the corresponding tests were based on compression, which makes ruptures of type (*d*) shown in Fig. 33 physically possible, supports the views suggested herein.

In the writer's opinion, this analysis confirms the masterful Terzaghi theory of "effective porosity," to be measured by experiments. Nevertheless, the writer is glad to state that the "Harza surface," conceived by the author, is very helpful in understanding the pore pressure problem.

[29] "Die Wirksame Flächenporosität des Betons," by Karl Terzaghi, *Zeitschrift des Oesterreichischen Ingenieur-und-Architekten Vereins*, Vol. 86, 1934.

[8] "Experiments on Effective Uplift Area in Gravity Dams," by S. Leliavsky Bey, in "Uplift Pressure in and Beneath Dams: A Symposium," *Transactions*, ASCE, Vol. 112, 1947, p. 444.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

EXPERIENCES WITH PREDETERMINING PILE LENGTHS

Discussion

BY H. CAMBEFORT

H. CAMBEFORT.[23]—Although the various examples given by Mr. Moore seem to confirm the design method in the paper, the writer is not completely convinced that this is so. It would have been helpful (especially for the European reader) if the theoretical basis for this method had been outlined. Apparently, the calculations depend on an assumption that there is passive earth pressure from the surrounding soil.

In the simple case of a wall, however, this pressure will not come into action before an appreciable displacement has occurred, and it is probable that the problem is more complex in case of a pile. Only experiments can supply an answer to this question; but it is necessary that these experiments be made with a certain objective insight, and some extra care. The usual test does not furnish serviceable information, giving only the penetrations as functions of the loads. This applies to most cases of piles driven in nonhomogeneous soil.

It is advisable for the engineer to examine the soil strata carefully, not only the layers which the pile must penetrate, but also those some distance below the pile tip. In general, a soft layer penetrated by a pile will eliminate the support that the upper and more rigid layers might give the pile. The upper layers, being subjected to the often important frictional forces along the pile shaft, cannot find support in the soft layer. The writer made a test on a pile that had been equipped with extensometers at the level of the boundaries of every soil layer. The stress in the pile immediately below the soft layer corresponded to nearly the total load applied at the pile top, whereas only one sixth of the total load was transmitted to the lower boundary of the upper layer (in the present problem). This distribution illustrates clearly the ambiguity of the theoretical case presented by Mr. Moore. Although the negative friction of all the layers seldom exceeds the resistance of the bearing

NOTE.—This paper by William W. Moore was published in November, 1947, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: March, 1948, by A. W. Farl; and June, 1948, by Hamilton Gray, Jacob Feld, and George F. Sowers.

[23] Civ. Engr., École Nationale des Ponts et Chaussées, and Head, Dept. for Studies of Constr., Entreprise de Fondations et Travaux Hydrauliques, Paris, France.

layers, Mr. Moore takes this friction into consideration consistently, without showing the consolidation diagram to justify it.

Furthermore, when it can be predicted that the upper layers will settle more than the tip of the pile (thus loading it as the result of the negative friction) the easiest procedure would be to eliminate this friction by lubricating the skin surface of the pile shaft. Thus, one should not try to drive the pile inordinately deep. Indeed, it is not always possible to do so.

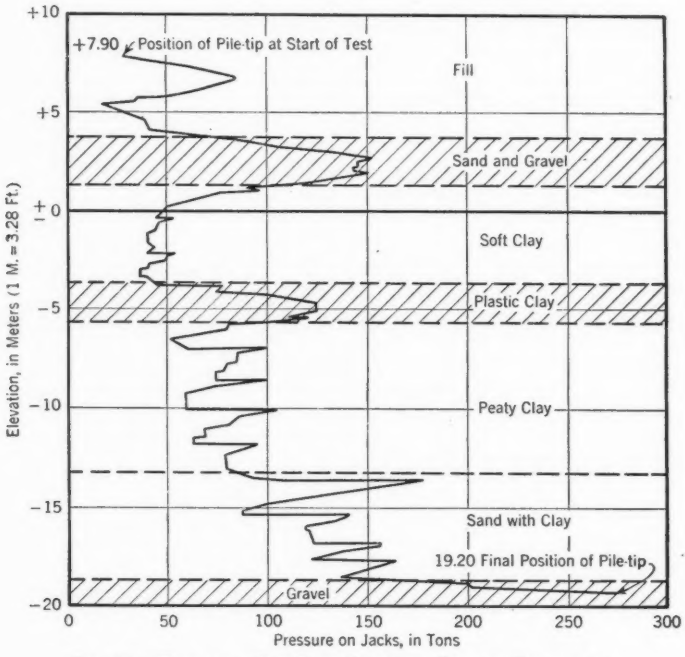


FIG. 22.—TEST OF A PRE-CAST CONCRETE PILE TO DETERMINE VARIATIONS IN ITS RESISTANCE TO PENETRATION

The soft layers above the resisting layer (the stratum in which the pile is supported) are also of great importance. If the pile tip is too close to the lower boundary of this resisting layer, and if this layer rests on soft soil, the pile will penetrate it—that is, will fail. On the other hand, if the pile had been shorter, the penetration would not have occurred and the pile would have met sufficient resistance. In such a case, however, settlement must be expected.

This phenomenon occurred in the first foundation for the Trans-Atlantic Railroad Terminal at Le Havre, France.[24] The piles, supported in the sand and gravel between El. +1.50 m and +4 m (from 5 ft to 13 ft), had sufficient bearing capacity, but the building settled nevertheless.

The foundations were repaired by jacking down pre-cast piles to El. −19 m (62.4 ft). The load-penetration diagram of one of these piles, as shown

[24] "Étude Expérimentale de la Force Portante d'un Pieu Foré," by H. Cambefort, Editions Science et Industrie, Paris, France, 1947.

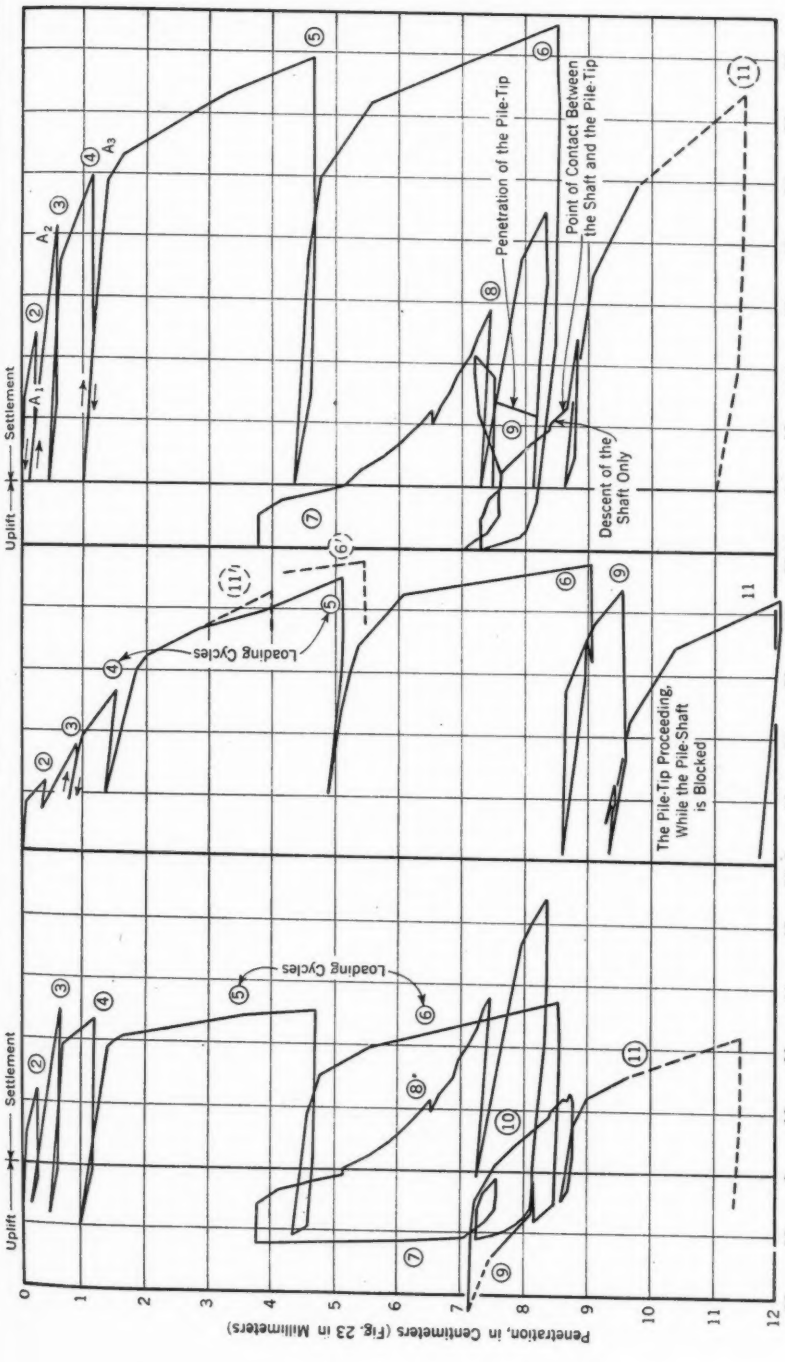

Penetration, in Centimeters (Fig. 23 in Millimeters)
Uplift — Settlement
Lateral Friction, in Tons
Loading Cycles
Load on Pile-Tip, in Tons
The Pile-Tip Proceeding, While the Pile-Shaft is Blocked
Penetration of the Pile-Tip
Point of Contact Between the Shaft and the Pile-Tip
Descent of the Shaft Only
Load, in Tons

FIG. 23.—VARIATION IN LATERAL FRICTION

FIG. 24.—PENETRATION OF THE PILE TIP

FIG. 25.—POINT OF CONTACT BETWEEN THE SHAFT AND THE PILE TIP

in Fig. 22, is a good illustration of the varying resistance.[25] When the tip reached the lower part of the firm layers (sand and gravels, and plastic clay) just above the soft layers, the bearing capacity decreased very rapidly; and, when the layer was penetrated, the bearing capacity was very nearly the same as it was above this layer. In other words, pressure of the soft strata has precisely the same effect as if the firm layers were not there.

Considering again the penetration of the firm layers, if the maximum capacity indicated in the diagram is created entirely by end bearing resistance, the bearing capacity of the pile will remain constant from the point where the pile tip reaches the soft clay. This is not the case. Before it stabilizes, the bearing capacity decreases through the first decimeters of this soft layer. It appears, then, that the lateral friction varies in the same manner as the end bearing resistance. This phenomenon was recognized in the test described herein. The pile tip was protected by a hydraulic jack, which enabled the investigator to measure the resistance.

Fig. 23 shows[26] that in this case (loading cycle 10), the lateral friction is much less prominent than in the case where the end bearing was active (loading cycles 3, 4, and 6). This curve shows also, that the lateral friction does not develop completely, except for a certain penetration of the pile (loading cycles 2 and 3). Furthermore, the frictional resistance to uplift is not comparable to the lateral friction generated by sinking the pile. However, it was approaching this value (corresponding to the settlement) when the end bearing resistance was omitted (loading cycles 7 and 10). Lifting the tip jack by inflation produces a stronger friction than lifting the pile by its head (loading cycles 9 and 7).

Thus, present computation methods are incapable of explaining all the phenomena. Therefore, it is necessary to collect experimental data. As a contribution to this end the writer submits the following: A rod, 130 mm in outside diameter, is driven into a very dense sand and gravel formation. When the rod reaches 5 m or 6 m, a crater is formed at the surface, about 50 cm in diameter (1 m is equal to 39.37 in. or 3.281 ft). The failure takes place along a vertical cylinder and the settlement reaches from 10 cm to 15 cm.

A tube 560 mm in diameter, driven in the same soils, produced failure within a cylinder 2.2 m (7 ft) in diameter. The ratio between the diameter of the disrupted soil prism and the diameter of the rod or tube was very nearly 4. Could this be regarded as an effect of the energy input at the tip? In a way, the actual pile was transformed into a hypothetical pile of much larger diameter, the proportion between the diameters being a function of the angle of internal friction of the soil. In the present case the proportion is 4, and the angle of internal friction is estimated at from 37° to 40°.

When the end bearing was zero, the lateral friction did not affect parts other than the actual pile shaft. In that case it would be four times smaller and, in any case, the small-scale tests made by J. Lehuérou-Kérisel[27] show

[25] "Étude Expérimentale de la Force Portante d'un Pieu Foré," by H. Cambefort, Editions Science et Industrie, Paris, France, 1947, p. 15, Fig. 17.

[26] *Ibid.*, p. 10, Fig. 12.

[27] "La Force Portante des Pieux," by J. Lehuérou-Kérisel, *Annales des Ponts et Chaussées*, May, 1939, p. 602.

that the frictional resistance F, to a pile driven into pure sand is given by the formula:

$$F = \frac{c\,s\,h}{2} \tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right) e^{(1.77+\phi)\tan\phi} \sin\phi \quad \ldots\ldots\ldots\ldots (15)$$

in which, in addition to the notation of the paper, c is the circumference of the pile and s is the vertical stress at a depth h. The accuracy of Eq. 15 was checked by tests on a pile of 420-mm diameter, the penetration being from 5 mm to 8 mm.

Eq. 15 should not be applied to cohesive soil. However, a sufficiently accurate approach may be made by simply multiplying the cohesion by the lateral contact area. The penetration of the tip is illustrated by a broken line in Fig. 24, where the changes of the slope correspond to the loads P given by the formula of J. Ferrandon:

$$P_1 = \frac{s\,A}{\frac{1}{2}\frac{1+\sin\phi}{\sin\phi}\sqrt{\frac{1+\sin\phi}{3-\sin\phi}} - 1} \quad \ldots\ldots\ldots\ldots (16a)$$

which is an extension of the formulas for piles devised by O. K. Froehlich[28] and by the formula of A. Caquot:[29]

$$P_2 = s\,A \tan^2\left(\frac{\pi}{4} + \frac{\phi}{2}\right) e^{\pi \tan\phi + 0.2\tan\phi} \quad \ldots\ldots\ldots\ldots (16b)$$

In Eqs. 16, A is the cross-sectional area of the pile tip. Unfortunately, the method of computing the corresponding penetrations t is not known. Possibly, the Boussinesq formula indicates a solution:

$$t = \frac{2}{\pi}\frac{s\,d}{a_v} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (17)$$

in which s is the stress on the tip of diameter d and a_v is a coefficient, depending on the soil. The analysis of this test shows that the penetration curve for the pile top (Fig. 25) presents[30] points of changing slope, corresponding to the following phenomena: A_1 is the limit of the elastic zone, the point defined by Messrs. Ferrandon and Froehlich; A_2 is the lateral friction now fully developed; and A_3 is the penetration and failure, the point defined by Mr. Caquot. Point A_2, Fig. 25, commonly referred to as the point of failure, corresponds to nothing more than the final development of the lateral friction.

These curves apply to a cast-in-place pile (drilled and cast in the soil), and it is possible that the conditions are somewhat different in the case of a driven pile. In particular, point A_1, Fig. 25, should disappear in the case of a driven pile. In so far as the designer does not consider a pile as an hyperstatic system, in which the resistance depends primarily on the displacements, he may conclude that the bearing capacity can never be calculated no matter what formula is applied.

---

[28] "Druckverteilung im Baugrunde," by O. K. Froehlich, Springer, Wien, 1934, p. 142.

[29] "La Force Portante des Pieux," by J. Lehuérou-Kérisel, *Annales des Ponts et Chaussées*, May, 1939, p. 583.

[30] "Étude Expérimentale de la Force Portante d'un Pieu Foré," by H. Cambefort, Editions Science et Industrie, Paris, France, 1947, p. 6, Fig. 3.

It seems to the writer that, in the case of cohesionless sands, investigators are "on the right track" with the Kérisel, the Ferrandon, and the Caquot formulas; but these formulas must be supplemented by a study of the displacements—a primary requirement for exploring the behavior of a pile. Next the two coordinates should be obtained for each of the points A_1, A_2, A_3, in Fig. 25. The same considerations may apply to the end bearing resistance in cohesive soils; but nothing is known concerning the lateral friction in such soils.

Mr. Caquot's "theorem of corresponding stages" reduces this case to one of a cohesionless sand, but it is only valid for a series of stabilized conditions of the soil, which does not at all correspond to the case involving a pile. In the same way, the theory of consolidation defined by Karl Terzaghi, Hon.M. ASCE, cannot apply to the first instant that a pile is loaded, because the phenomena occur much too rapidly at that stage. It can only apply when data are available as a result of a study of the life of the foundation.

The structure of clay resembles that of sand, but it is likely, that their physical properties are more like those of solids or liquids of high viscosity, than those of sands. It seems more convenient, however, to abandon consolidation and shear tests, and to devise others, with sufficiently reasonable similarity to the actual conditions of a pile (or of the failure of a soil).

In order to do this, it may be useful to treat the soils as perfectly plastic media, or even as viscous liquids, and then to study their "seepage" and their "yield value." The experience of the glassmaker should not be forgotten. It may be interesting, from another point of view, to determine the apparent coefficient of elasticity of soils more frequently. Such studies, in connection with tests, will undoubtedly help to clarify the problem of determining the bearing capacity of piles.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

SHEARING STRESS DISTRIBUTION IN BOX GIRDERS WITH MULTIPLE WEBS

Discussion

BY A. S. NEIMAN

A. S. NEIMAN,[15] ASSOC. M. ASCE.—Discussers of the paper have contributed good derivations of formulas, have singled out certain limitations of the theory, and have expanded the solution to the more general cases involving torsion. Professors Clark and Hanson are especially to be commended for using arithmetic to show how the distribution of shear stress is obtained when torsion, in addition to bending, is to be considered.

Professor Goldberg's attention is drawn to the first sentence under the heading, "Acknowledgments," where the following is stated:

> "The procedure for computing shear stress distribution and the particular application of the principles of mechanics to the solution of the problem as outlined in this paper were developed by the writer while engaged in the structural design of concrete barges for the United States Maritime Commission."

No claim is made that principles of mechanics are developed. Professor Goldberg's statement that this problem is practically routine in the aircraft industry is undoubtedly true. Unfortunately, however, the large majority of structural engineers have not had experience in aircraft design. Moreover, as shown by Professors Clark and Hanson, the literature pertaining to civil engineering structures is somewhat scant in this regard. The problem was also complicated by the meager information available as to details for resisting shearing stresses at transverse frames, at transverse bulkheads near (and in the way of) large hatches, at corners, and at abrupt changes in thickness of plating. These subjects may furnish a field for further study and experimentation.

NOTE.—This paper by A. S. Neiman was published in February, 1948, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: June, 1948, by John E. Goldberg; and September, 1948, by J. G. Clark and W. E. Hanson.

[15] Chf. Structural Engr., Vacuum Concrete, Inc., Philadelphia, Pa.

Full-scale hogging and sagging tests on concrete and steel ships were conducted by the United States Maritime Commission during 1943–1945. The ships were floated in still water and loaded by filling various holds with water ballast. Electrical strain gages in the form of rosettes were placed all around the girth of the vessel near points of maximum moment and shear. The vertical shearing stresses were computed from the strains. It will be of interest to engineers to know that the vertical shear computed to exist at the section from the diagram of loading checked within 10% of the total of the internal shearing forces as computed from the strains.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

PANAMA CANAL—THE SEA-LEVEL PROJECT A SYMPOSIUM

Discussion

BY ERNEST SHANKLAND, AND JOHN S. MCNOWN

ERNEST SHANKLAND.[126]—In the fifth Symposium paper, Messrs. Lee and Bowers cite the paper by G. S. Baker[33] as one of several sources of information on ship resistance in restricted channels (see heading, "Studies of One-Way Traffic"). Using Mr. Baker's studies as a background, the writer investigated the maneuvering factors in the Thames tideway in England, which is a very restricted waterway considering the trade that it carries. Mr. Baker's laws of safe steering were found to be rather conservative. Observing the starboard hand rule, it is sometimes possible within Mr. Baker's limits to have several vessels navigating the bends at high tide, with a stream fluctuating in force and direction.[127] By experimentation when designing new dredged cuts it was discovered that curves should be held within a 10° limit, to control course alteration, and to retain a channel having parallel sides during the dredging; and, as advocated by Messrs. Lee and Bowers, half-moon concave additions should be avoided at the bends, as they tend to pull the convex side over to them.

JOHN S. MCNOWN,[128] Assoc. M. ASCE.—Perhaps the most striking of the several significant features of ship behavior in canals described by Messrs. Lee and Bowers is the change in level with speed indicated in Fig. 50. Although the measured changes in level of several feet are surprising on first consideration, they are readily confirmed—quantitatively as well as qualitatively—by the

NOTE.—This Symposium was published in April, 1948, *Proceedings*. Discussion on this Symposium has appeared in *Proceedings*, as follows: June, 1948, by Hans Kramer, and Philip G. Nichols; September, 1948, by George B. Pillsbury, Kenneth S. M. Davidson, Joel D. Justin, W. H. McAlpine, W. E. R. Covell, William Herbert Hobbs, Hibbert Hill, Kenneth C. Reynolds, Gregory P. Tschebotarioff, Charles W. Dohn, and Donald F. Horton; October, 1948, by E. Montford Fucik, Charles M. Romanowitz, and Raphael G. Kazmann; November, 1948, by George B. Massey, William Allan, and Boris A. Bakhmeteff; December, 1948, by Robert C. Sheldon, F. W. Edwards, and H. R. Cedergren; and January, 1949, by Harry O. Cole, Ole P. Erickson, Clarence S. Jarvis, Anson Marston, and Ralph Z. Kirkpatrick.

[126] Assessor and Surveyor, Folkestone, England.

[33] "Steering of Ships in Shallow Water and Canals," by G. S. Baker, *Transactions*, Institution of Naval Architects, Vol. 66, 1924, pp. 319–340.

[127] "Modern Harbours: Conservancy and Operations," by E. C. Shankland, James Brown & Son, Glasgow, Scotland, 1926.

[128] Associate Prof., Dept. of Mechanics and Hydraulics, and Research Engr., Iowa Inst. of Hydr. Research, State Univ. of Iowa, Iowa City, Iowa.

straightforward application of elementary concepts. In order for the familiar form of the Bernoulli equation to apply, the flow pattern must be changed from unsteady to steady by the simple superposition of a uniform flow equal and opposite to the motion of the ship. In this way the ship is brought to rest and the water in the canal is given a steady motion as shown in the two parts of Fig. 115. The resulting pattern of flow then differs only in minor detail from that for flow in an open channel with bottom or side contractions. The reduction in cross-sectional area leads to an increase in velocity which, in tranquil open-channel flow, must be accompanied by a drop in surface level similar to that occurring over a broad-crested weir or in a Parshall flume.

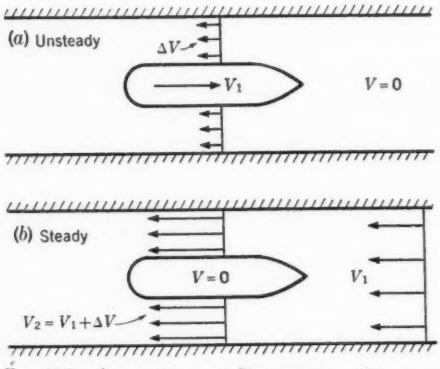


FIG. 115.—APPLICATION OF PRINCIPLE OF RELATIVE MOTION TO SHIP AND CANAL

Because the changes in level seemed large to the writer, representative calculations based on the aforementioned concepts were made. Such remarkably close agreement was found that the computations were completed for nine cases shown in Fig. 50, the assumption being made that the ship is long enough for the flow past it to be essentially uniform throughout the channel cross section along the entire length of the ship. Although the change in water surface is surely less than average near the bow and stern, the effect on the buoyancy is also less in these regions, and no correction is necessary to estimate the effect in question. Although Fig. 49 serves as an approximate check on this assumption, the reversed flow past the ship for the conditions of Fig. 49 is so nearly critical in the reduced section that complete uniformity could scarcely be expected. In one respect the application of the usual one dimensional form of the Bernoulli equation is more accurate for this case than in the usual applications to open-channel flow—the velocity in the approaching flow is actually, instead of approximately, constant over the entire cross section.

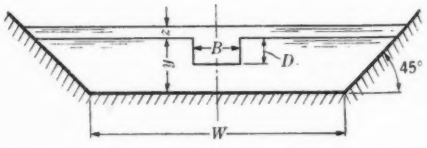


FIG. 116.—DEFINITION SKETCH FOR FLOW CROSS SECTIONS

For steady flow, with the water moving and the ship stationary, the velocity of the undisturbed water V_1 is equal to the ship speed, whereas the velocity in the reduced section V_2 is somewhat greater as shown in Fig. 115. The change in level, which may be designated as z, is simply related to the two velocities,

$$z = \frac{(V_2)^2}{2g} - \frac{(V_1)^2}{2g} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

Also, from the continuity equation written for the two sections shown in Fig.

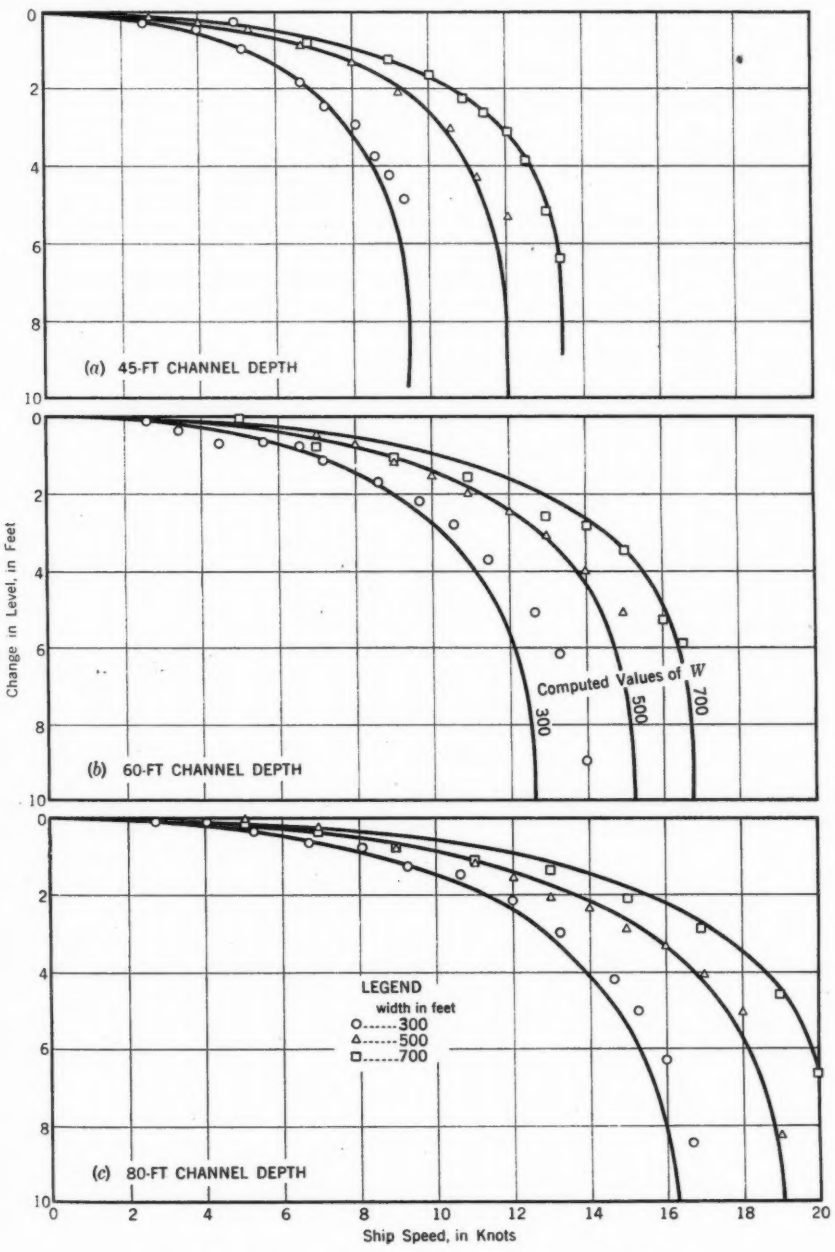


FIG. 117.—COMPARISON OF COMPUTED AND MEASURED CHANGES IN LEVEL AS A FUNCTION OF SHIP SPEED
(Ship and Model Data Are the Same As Those in Fig. 50)

116, a second relationship can be obtained:

$$\frac{V_1}{V_2} = \frac{A_2}{A_1} = \frac{(W + y - z)(y - z) - B D}{(W + y)(y)} \quad \ldots\ldots\ldots\ldots (8)$$

in which B is the width, or beam, of vessel; and D is the depth of vessel displacement. From Eqs. 7 and 8 the variation of V_1 with y can be computed directly for assigned values of the beam (B) and draft (D) of the ship and of the depth and the width of the channel. The area of the submerged cross section was arbitrarily assumed to be $110 \times 32 = 3{,}520$ sq ft rather than $113 \times 32.25 = 3{,}650$ sq ft, since the shape is not a true rectangle. The computed curves are shown in Fig. 117, together with the experimental points scaled from Fig. 50. The agreement between the results from the analysis and those from the model study is remarkably good.

Both the theoretical curves and the measured values indicate the existence of a maximum velocity, which, as the authors have shown, is that which leads to critical flow in the reduced section. Once this maximum is reached, the change in depth becomes unstable since the slope of the ($V - y$)-curve is infinite. If an even higher speed were attempted, water would gradually pile up ahead of the ship. The critical velocity, therefore, should be used as a design maximum, which should not be closely approached and surely never exceeded.

The simplified analysis utilized herein is restricted to cases in which the channel dimensions and the ship dimensions are comparable in magnitude. Motion of a ship in a large river, or in the ocean, could not be analyzed in this manner since the increases in velocity would be large near the ship, decreasing to zero some distance away laterally, in contrast to the constant velocity assumed in the derivation of Eqs. 7 and 8. For a very large canal cross section the change of surface elevation near the ship is thus a function of the local flow pattern rather than of the canal dimensions. The analysis for this limiting case would be extremely complex even if the induced surface wave pattern were neglected; observations of the magnitude of the comparatively small effect are doubtless available from model tests.

The change in surface level near a ship, and hence the change in ship level as well, are readily explainable and also predictable in magnitude on the basis of a simplified analysis. Although limitations on the applicability of the proposed analyses are evident, within the range of practical canal design, surprisingly good results can be obtained. Logical conclusions that can be drawn from the comparison in Fig. 117 are that further model studies in the range of variables tested are unnecessary, and that many of the runs made could have been omitted once agreement had been established for a few representative conditions. Since the theoretical curves in Fig. 117 are easily prepared, prediction of given ship and channel dimensions of the variation of surface level with speed, together with the critical or maximum ship speed, is reduced to a simple procedure.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

VALUATION AND DEPRECIATION OF PUBLIC UTILITY PROPERTY

Discussion

BY ANSON MARSTON, AND JAMES P. MCKEAN

ANSON MARSTON,[39] PAST-PRESIDENT AND HON. M. ASCE.—The part of this paper most valuable to the writer is the discussion (under the heading, "Introduction") of the indications of the present views of the United States Supreme Court on the authority of the 1898 Smyth versus Ames decision—as shown by the recent rulings in the Natural Gas Pipeline Company Case (315 US 575),[6] the Hope Natural Gas Case (320 US 591),[7] and the Panhandle Eastern Case (324 US 636).[8]

In brief, the essential principle of the Smyth versus Ames decision is simply that the "fair values" of utility properties must be determined in the same way as all other property values are determined in sales and purchases—by the judgments of the parties thereto. Every factor which should affect such judgments must be considered and these factors given their "just and right" weights in each case—as determined by sound judgment, not by any formula. Instead of Mr. Hale's [12] Does the Ghost of Smyth versus Ames Still Walk? the true present question is: Does the Soul of Smyth versus Ames Still Survive? To this the writer would answer yes—and not even the Supreme Court can destroy this true basis of every exchange value transaction, any more than it can destroy the law of gravity. Whether this long-established principle still prevails with the present Supreme Court is anybody's guess—as the author of the paper correctly concludes.

The rest of the author's discussion on this point is devoted to presenting what the writer considers to be an unsound and unwise "suggestion." It is the suggestion that, in valuations of utility properties and determinations of

NOTE.—This paper by Maurice R. Scharff was published in June, 1948, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: October, 1948, by J. Kappeyne; November, 1948, by W. V. Burnell; and December, 1948, by Donald Gunn, Henry F. Lippitt, II, and Albert P. Learned.

[39] Dean Emeritus of Eng., Iowa State College, Ames, Iowa.

[6] *Public Utilities Reports*, New Series, Vol. 42, 1942, p. 129.

[7] *Ibid.*, Vol. 51, 1944, p. 193.

[8] *Ibid.*, Vol. 58, 1945, p. 100.

[12] "Does the Ghost of Smyth v. Ames Still Walk?" by Robert L. Hale, *Harvard Law Review*, Vol. 55, 1942, p. 1116.

their accrued and annual depreciations, a return be made to the Smyth versus Ames decision, but that only two sentences in it be used, discarding its requirements that every factor affecting value must be given due consideration; and that, in adopting an interpretation of these two sentences, a valuation method be used in which everything is based on calculated estimates of the reproduction cost of existing properties and of the reproduction cost of substitute properties, designed but not built.

The Smyth versus Ames "fog" of meaning, which various economists, commission authorities, lawyers, some judges, and other writers on valuation find so blinding, is in their own minds. If in every purchase and sale of property the exchange value had to be determined in court, the same "fog" would ensue. It arises from the conflicting testimony of experts, paid by opposing parties in court proceedings. Each such party can be sure that it can hire a battery of "experts" to support its particular contentions. The "suggestion" made by the author of this paper would, if adopted, make his present "confusion worse confounded" even more confounded.

It is the writer's view that the trend of progress in valuation and the determination of depreciation is in the opposite direction from that suggested by the author. More and more, there is a departure from the estimates, and sometimes fanciful speculations, involved in experts' opinion testimony in court cases.

Instead, the trend is to give more and more weight to fully itemized property inventories showing the actual original costs of the individual items of the actually existing properties; and to only such overhead percentages of cost as have been actually incurred in the past; but not excluding such substitution of present prices (and the resulting cost values of individual items) as sound, conservative judgment deems certain to prevail for a considerable future period.

In the opinion of the writer, the same methods of estimating the values and the depreciations of public utility properties should be used as are applied to other properties.

JAMES P. MCKEAN[40] M. ASCE.—In view of the rapidly spiraling construction costs and the increasing demand of American industry for additional plant facilities this paper is a timely one. The proposed method of valuation is presented as a tool to clear the "confusion worse confounded" in which the subject of valuation and depreciation is now enmeshed. This writer wishes to direct his discussion toward some of the underlying causes of this confusion and to suggest certain criteria that must be satisfied by any method of valuation that is to be generally accepted as the better measure of value.

Methods of Pricing.—Common methods in use for pricing manufacturing inventories include the "first in-first out," "last in-first out," "accumulated average cost," "cost or market, whichever is lower," and "standard costs." No one of these methods has universal acceptance. Pricing property inventories is a specialized application of the general pricing problem. In pricing inventories for the purpose of valuation the two more widely accepted methods are "bottomed" on either (1) original construction costs or (2) present con-

[40] Prof., Dept. of Gen. Eng., Iowa State College, Ames, Iowa.

struction costs. Both methods have their variations, and are adjusted to reflect the accrued depreciation on the units in service.

Replacement Cost Experience, Past and Present.—The desirability of using the present cost of construction as a basis for value was vigorously and forcefully presented in the famous Smyth versus Ames Case (169 US 466). Labor was scarce following the Civil War, financing was difficult to obtain, the United States was in a period of postwar inflation, working conditions on the transcontinental railroads were rugged, and the resulting original costs for these arteries of commerce were high. For three decades prior to the Smyth versus Ames litigation construction costs had been decreasing. Such was the cost setting at the time the representatives of regulation argued so vehemently for the use of the "present costs of construction" as the criterion of value. The United States Supreme Court held, however, that all factors had to be considered.

During the following two decades construction costs remained relatively constant, but with the outbreak of hostilities in Europe, heralding the beginning of World War I, construction costs again advanced. They "peaked" in 1920, dipped in 1921, and then leveled out into a plateau for the remainder of the 1920's. As these costs rose, and particularly after they had been temporarily stabilized on a new level, the proposal to use present reproduction costs as the primary, if not the only, measure of value was given favorable reception, and often accorded major weight in the "fair values" that were found.

The U. S. Supreme Court handed down a decision in 1930 holding that the legal depreciation base should be "present value," and not cost in the case of the United Railways and Electric Company of Baltimore versus West (280 US 234). Economic conditions changed markedly thereafter, and with the depression of the early 1930's this case was avoided by the Court in their decisions handed down in later rate cases.

By 1944 economic conditions showed another change, and in the Hope Natural Gas Case (320 US 591) the Court specifically overruled its 1930 decision declaring, "we cannot approve a contrary holding in the United Railways vs West."

With the advent of World War II, and particularly after the lifting of price controls, there was another substantial rise in construction costs with the attendant economic problems. Mr. Scharff has proposed the use of the "replacement cost of the most economical substitute plant" as the desirable measure of value. A method of estimating value and depreciation somewhat similar to this proposal is used by the United States Steel Corporation. In its publication[41] the annual depreciation charge is shown in two parts—$43,600,000, based on the original cost of the facilities, and an additional $13,100,000, covering wear and exhaustion of facilities "in view of the greatly increased present cost of facilities to be replaced." In a later edition of the same publication[42] the method is again mentioned with the explanation that it is used:

---

[41] *U. S. Steel Quarterly*, August, 1948, p. 1.

[42] *Ibid.*, November, 1948, p. 1.

"* * * in an endeavor to set up a reserve for wear and exhaustion of facilities which would more realistically accord with such higher current costs, an additional charge to cover more accurately such current wear and exhaustion was made."

Other current literature reflects some of the same thinking in discussing depreciation and replacement problems.

Original Costs and Price Fixing.—The original cost of construction of a property has long been used as one factor to be considered in determining value. Since before 1898 price fixing has been a recognized prerogative of the legislature (Munn versus Illinois (94 US 123) in October, 1876). Rate making is but a species of price fixing. In passing the Natural Gas Act Congress was exercising the right to fix prices when it vested the Federal Power Commission with the responsibility of determining the "lowest reasonable rate." With such a mandate it is not surprising that those concerned with administering the Natural Gas Act should use the original cost approach, when such costs are lower than the cost levels at the time of the rate controversy.

One of the most recent Supreme Court decisions which bears on the matter of price fixing is the United States versus John J. Felin and Company (68 US 1238) decided on June 14, 1948. In this case the government had requisitioned meat products belonging to a meat slaughterer under the authority of the Second War Powers Act. The food distributing authority offered compensation based on the existing ceiling prices of the Office of Price Administration (OPA). The slaughterer refused to accept the offered compensation and sued for "just compensation" under the Fifth Amendment in the Court of Claims. That court decided that the replacement cost of the meat products, and not the maximum ceiling price, was the proper measure of value.

An appeal was taken to the U. S. Supreme Court which reversed the Court of Claims. Three justices (Chief Justice Fred M. Vinson and Justices Felix Frankfurter and Harold H. Burton) held that the "replacement cost" basis utilized by the Court of Claims was spurious, and that the burden was on the slaughterer to prove that the compensation offered by the government was not "just." In their view the slaughterer had failed to meet this burden. Three other justices (Justices Stanley F. Reed, Hugo L. Black, and Frank Murphy) concurred separately in the judgment on the ground that, under the circumstances of the case, the constitutionally established maximum price (OPA ceiling) should be considered "just compensation" within the meaning of the constitution.

Justice Wiley B. Rutledge concurred in the opinion. He advanced the reason that in the case of government requisition of perishable commodities the legal or ceiling price should furnish presumptively the measure of just compensation, subject to revision if the owner sustained the burden of proving he was entitled to a greater reward. Although these seven justices advanced different lines of reasoning, they united in opposing the use of "replacement cost" as the basis of value.

Two justices (Justices Robert H. Jackson and William O. Douglas) dissented, holding that:

> "* * * the constitutional guaranty of compensation * * * becomes meaningless if the government may first, under its 'war power' fix the market price and then make its controlled figure the measure of compensation."

This case is a recent pronouncement of the Court and indicates its opinion on the use of "replacement cost" as a measure of value in pricing meat confiscated by a governmental agency.

Governmental bodies have been building for themselves a place in the management of the economic affairs of the United States since the passage of the Interstate Commerce Commission Act in 1887. At first their growth was incipient; but with the legislation of the 1930's and the extensive powers vested in such agencies as the Securities and Exchange Commission, the Federal Communications Commission, and the Federal Power Commission, the relationship between government and business has ripened into a close association. Uniform systems of accounts, and property records developed on the original cost basis, have served to make precise cost records readily available. The legislative mandate in the creation of these commissions and the availability of the facts based on original cost have resulted in many decisions decidedly favorable to the original cost philosophy. In favoring the use of the original cost basis, Everett C. McKeage[43] suggests:

> "The advantage of the cost formula as contrasted with other formulas is that it is certain or can be made certain. * * * It appeals to me that the very nature of a public utility, occupying as it does a quasi-public and a quasi-trustee position, lends itself to the application of a rule of valuation in consonance with the obligations flowing from such a public status. It would appear that valuation on a cost basis fits into such a status. * * * I believe that the cost basis is preferable for valuing all property for rate fixing purposes."

The Administrative Procedure Act passed in 1946 (Public Law No. 404, Seventy-ninth Congress) has served, and will continue to serve, to clarify the position of the governmental regulatory body. The Felin decision antedates the Administrative Procedure Act and must be noted for the philosophy it develops as well as its timing. It is evident that one who proposes the use of reproduction cost or "replacement cost" must meet the original cost theory point by point, and in order to prevail must show a definite superiority.

When Is a Public Utility Property "Devoted to Public Service"?—In selecting the more desirable price basis it is necessary to determine the time at which the unit of property is "devoted to public service." The "replacement cost" theory is predicated upon the idea that the title to the units of property remains in the owners, and that as the small increments are actually "consumed" in service they are then considered to be "devoted" to the public service.

Some of the advocates of the original cost approach believe that at the time the property is purchased, installed, and placed in service it immediately is "sold" to the public. The cost at the time of construction measures the value when new, and the resulting charges made for depreciation are merely prepaid

[43] "Public Utility Property: Views of Commission Counsel as to Valuation," *Journal*, Am. Bar Assn. December, 1948, p. 1156.

expenses for the cost of the property to be returned to the owners over the period selected for charging off such depreciation. This line of thought parallels that of the investor who places his money in the debt obligations of utility companies. This investor expects to be paid interest at fixed intervals, and to have his principal returned to him at the maturity date. The investor anticipates no adjustment in either principal or interest because of variations in the purchasing power of his money which is used to purchase the property units serving the public.

Although the Court spoke repeatedly on this point some years ago, recent clear-cut pronouncements since the appearance of some of the new regulatory agencies are lacking.

TABLE 3.—VALUATION BASIS FOR AN ORIGINAL AND REPLACEMENT COST ANALYSIS

Original cost	Present construction cost of the most economical substitute plant
	(*a*) PRICING OF UNITS
Facts from actual construction cost records of the units in service	Part: Facts from cost records for units now in service where the construction cost levels have not changed. Part: Estimated present construction costs for installed units which are considered the most modern to operate. Part: Estimated present construction costs for the estimated most economic substitute units.
	(*b*) OPERATING COSTS
Actual experience of units in service projected into the near future	Part: Actual experience of units now in service, projected into the near future. Part: Estimated costs on estimated substitute units not yet installed or operated by this plant.
	(*c*) INVENTORY OF UNITS
Actual items now installed and operating	Part: Usable units now installed and operating. Part: Estimated substitute units.
	(*d*) DETERMINATION OF PROBABLE LIVES
From best data of actual experience and mortality characteristics of units retired and now in service	Part: From best data of actual experience and mortality characteristics of units retired and still in service. Part: Estimated lives on estimated substitute units.
	(*e*) WHEN IS THE PROPERTY DEVOTED TO PUBLIC SERVICE?
Construction date is definite	Time of consumption is a matter of estimate.

Facts and Estimates.—In addition to the problems of pricing and time when a property is devoted to public service, other elements in a valuation include: (1) A determination of operating costs, (2) a determination of the proper units of property to inventory, (3) selection of the probable lives of the units for

depreciation purposes, and (4) selection of the depreciation method. Table 3 indicates the source material available, and the amount of estimating required, for original and replacement cost analyses.

Information and Personnel Available for Making Intelligent Replacement Studies.—In the early days of American industry reliable records of construction costs were notable by their absence. In the early part of the twentieth century considerable weight was accorded estimated costs of reproduction because original costs were not available, nor was there substantial variation between original and reproduction costs. Fact is generally considered preferable to estimate, and where estimates have been used to influence the finding of value "proof of fair value must be full and convincing" as in the Missouri rate cases, Knott versus the Chicago, Burlington and Quincy Railroad Company (230 US 474) in June, 1913.

Estimates will be recognized as evidence if well supported, but the Court is likely to look with a jaundiced eye on the basic assumptions as it did in the case of the United Fuel Gas Company, et al, versus the Railroad Commission of Kentucky, et al.[44] It stated (concerning some inconclusive estimates) that:

> "* * * such predictions can only be made on the basis of data which are not and cannot be known, and most of which are in the highest degree speculative. Such process of estimating value is without any known sanction."

Although economic analyses can be made with reasonable precision, they involve careful study and require time, money, and data. The requirements for making estimates of the most economical substitute plant involve all the hazards of estimating the present construction costs of the present plant and add the requirement of a proper selection of the most economical substitute. Such estimates vary from rough approximations to precise estimates.[45] Eugene L. Grant,[46] M. ASCE, suggests that "the burden of proof should always be on the proposed replacement." Professor Grant also indicates[47] that obstacles such as unconscious bias, personal financial interest, errors in costs or estimates, and incorrect estimates of economic lives of proposed machines and structures are ever present. In his closing paragraph, he states:[48]

> "* * * as a general rule, an estimator cannot be expected to make a successful estimate for a structure he would not know how to build. * * * Engineers are frequently called upon to make preliminary estimates of the cost of work with which they are not entirely familiar. Although many such estimates must be made, the estimator should realize the resulting danger of errors."

Certainly it is true that contractors have their own estimators to guide them in their bids. These estimators are specialists and in relatively short supply. The writer's experience and observations, since about 1928, lead him to the

[44] *Public Utilities Reports*, Vol. 1921A, p. 433.

[45] "Engineering Economic Analysis," by Clarence E. Bullinger, McGraw-Hill Book Co., Inc., New York, N. Y., 1942, Chapters 6 to 8.

[46] "Principles of Engineering Economy," by Eugene L. Grant, Ronald Press, New York, N. Y., 1938, p. 206.

[47] *Ibid.*, p. 363.

[48] *Ibid.*, p. 405.

conclusion that most of the concerns in the middle west would not have sufficient information or an adequate staff to make the economic studies for their entire properties. They might find it a substantial burden to recruit, train, and maintain sufficient personnel to make these estimates on a company-wide basis. Many of the operating companies do have engineering and economic staffs engaged in making economic studies; but they are limited to the older property and those sections of plant that are unique and require special analyses. Since Mr. Scharff's proposal depends on the precision of many estimates, special efforts need to be made to convince all parties interested in valuation that adequate records and sufficient technical personnel are available to do an eminently satisfactory job.

A study[49] released in 1948 is concerned with this replacement problem and reports information on depreciation and replacement policies of almost two hundred different machinery manufacturing companies whose financial well-being is vitally affected by this problem. The summarization of part of their findings is interesting, as presented in Table 4.

TABLE 4.—Depreciation and Replacement Policies of Machinery Manufacturing Companies (Percentages of Total Companies Reporting Yes or No)

Item	Companies having provision for:	Yes	No
1	An equipment engineer that specializes in making replacement studies	28	72
2	Making periodic reviews of their equipment policies	35	65
3	Necessary collateral records of problems affecting replacement, such as spoilage of work, machine hours per year, and lost time for repairs	34	66

It is recognized that these data are not for the utility industry; nevertheless they do indicate a paucity of trained personnel and records necessary for making intelligent replacement studies in the machinery industry.

Aids to Clarification.—The field of valuation touches the entire national economy; it is not the exclusive province of the civil engineer, nor of the engineering profession. Accountants, architects, business executives, economists, financiers and investment bankers, lawyers, and public administrators—as well as many others—are vitally concerned with some or all the phases of valuation. Any acceptable method must be (*a*) capable of precise determination by qualified personnel; (*b*) reliable in application so that qualified valuators, even though representing hostile financial interests, will reach substantially equivalent results; (*c*) susceptible of proof; (*d*) appealing to reason and justice; and (*e*) understandable in principle to others, not only to the professional practitioners. These other parties are also concerned as to the source, the quantity, and the quality of the available personnel to expedite any proposed method of valuation. The necessary training required for new entrants into the field, and their ability to produce consistent results at reasonable cost, are also matters of interest.

The valuation engineer can assist in clarifying the basic principles by so preparing and presenting his findings that separate decisions become mandatory

[49] "MAPI Survey of Replacement and Depreciation Policies," *Bulletin No. 2119*, Machinery and Allied Products Inst., Chicago, Ill., October 1, 1948.

on (1) the proper units to include in the valuation inventory; (2) the proper pricing basis; (3) the acceptable methods of determining probable lives to be used in depreciation calculations, especially for the proposed units to be incorporated in the substitute plant analysis; (4) the depreciation methods to be used to calculate both annual and accrued depreciation; and (5) the time when the "property is first devoted to public service."

The Society has long been in the forefront of progress in the valuation field and has had a Joint Technical Division Committee on Depreciation, as well as a Committee on Valuation Procedure in the Engineering Economics Division. When the notable and comprehensive "Report of the Committee on Depreciation" of the National Association of Railroad and Utilities Commissioners (NARUC) was published, in 1943[50] a special committee of the Board of Direction, ASCE, was appointed to prepare an immediate reply.[5] This small committee was only partly representative of the entire Society membership; it was working under severe time limitations and the report, "Principles of Depreciation," was disappointing except for some excellent dissenting discussions. It appears to be the proper province of the constituted committees of the Society to make continuing studies on this subject and present to the membership carefully considered progress reports.

For several years the writer has waited for an expression from these committees or the members thereof to clarify points developed in the NARUC report or in Mr. Scharff's paper. It is hoped that Mr. Scharff's dissertation will serve to reactivate new Society committees so selected as to be truly representative of all segments of the Society membership, so that notable solutions to some of the present problems can be had and so that ASCE can resume its proper place in the field of valuation.

Mr. Scharff has rendered a distinct service by so presenting his proposals that the issues are joined. The decision of the Supreme Court on the appeal of the Attorney General of the United States will be read with interest in the hope that it will help chart the course of valuation in the light of present economic conditions.

[50] "Report of Committee on Depreciation," National Assn. of R. R. and Utilities Commrs., Washington, D. C., 1943.

[5] *Transactions*, ASCE, Vol. 111, 1946, pp. 1343–1400.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

FLEXURAL CONSTANTS OF HAUNCHED BEAMS BY AREA COMPUTATION

Discussion

BY ROBERT V. HAUER

ROBERT V. HAUER.[17]—The method presented by Professor Tsai for analyzing haunched beams (in fact, beams of any shape) is an interesting contribution to the theory of flexure in so far as it shows that it is possible to derive the shape constants and the load constants for all common loading conditions from the reciprocal I-diagram and its successive momental areas alone. As the same problem can be solved in several other ways, the relative merits of this new method must be judged by its simplicity and its accuracy.

The title of the paper does not quite reveal the characteristic feature of the method described; for, as every end angle change of a beam is represented by a definite integral, the evaluation of such an integral, if not done directly, is always (in more or less concealed form) the computation of an area. It is only the special arrangement of the various area computations that characterizes the author's method. Professor Tsai's procedure is based on the fact that the moments for all common loading conditions (including those connected with the shape constants) are whole algebraic functions of the distance x from one end of the beam. This makes it possible to compute first the component areas defined by the successive powers of x in the integrand, instead of directly evaluating the areas representing the various end angle changes. By combining these components, or "momental," areas properly, the required angle changes are found.

This procedure would result in a simplification only in the exceptional case when the number of component areas was smaller than the number of required angle changes. For the three angle changes representing the shape constants three momental areas must be evaluated (A, A_1, and A_2). One more (A_3) is required for the two angle changes that occur under a uniformly distributed load; but this case can also be solved directly with only one additional opera-

NOTE.—This paper by Fang-Yin Tsai was published in September, 1948, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: December, 1948, by J. Edmund Fitzgerald, and Harris Solman.

[17] Structural Engr., Albert Kahn, Inc., Detroit, Mich.

tion, because it follows from Eqs. 37 and 19 that

$$\alpha^{o}_{o(2)} + \alpha^{o}_{n(2)} = \tfrac{1}{2}\,\beta\, w\, L^2 \ldots\ldots\ldots\ldots\ldots\ldots (145)$$

—a relation which is a generalization of Eq. 43. If a concentrated load acts on the beam, three new areas have to be evaluated—namely, $A_{(kn)}$, $A_{1(kn)}$, and $A_{2(ok)}$—and a uniform load covering part of the beam requires the computation of even five additional areas, $A_{(kn)}$, $A_{1(ok)}$, $A_{2(ok)}$, $A_{1(kn)}$, and $A_{3(ok)}$, whereas for each of these loading conditions the two load constants can be found directly in two operations. Only for a triangular load can a saving be achieved by the author's method, as only one additional area (A_4) is required in this case.

It must be added that, when partial momental areas are to be evaluated, it will often be necessary to divide the beam into a greater number of sections. For instance, in Example I, if a concentrated load were assumed to act at $x = 0.1\ L$, the division of the haunched part of the beam into six sections would evidently not be sufficient.

Finally, considering that, after the various areas are found, the task remains to combine them properly, the writer doubts if the method suggested by the author really excels in simplicity.

The process of plotting and evaluating the I_o/I-diagram and its momental areas can be considered a preparatory work intended to facilitate the computation of the angle changes. That this does not result in a greater simplification is explained by the fact that the preparatory work must be repeated for each beam of individual shape. The writer believes, therefore, that Professor Tsai's method cannot compete in convenience with another method of computing angle changes, a procedure which, like the author's method, is based on some preparatory work eliminating the necessity of plotting the various M/I-diagrams, but has the great advantage that the preparatory work needs to be done only once.

Briefly outlined, this scheme consists of dividing the beam into a number of equal parts (say, ten) and attributing to each section a constant reciprocal moment of inertia equal to its average value in this part. If this is done, the integral representing an angle change for a certain loading condition can evidently be expressed as a sum of ten terms with numerical coefficients that can be evaluated and tabulated once and for all and used for a beam of any shape. The actual computation of an angle change then simply consists of multiplying a set of tabulated figures with the respective I_o/I-values of the beam in consideration and summing up, which is a very convenient and relatively time saving operation.

If reasonably applied, all methods of computing the end angle changes of a haunched beam are sufficiently accurate from a practical standpoint. It must be borne in mind, however, that the basic integral formula for the angle changes is itself strictly correct only for a prismatic beam. When applied to a beam with variable cross section, even if the integration is performed with mathematical exactness, it leads to a result that is only a physical approximation.

In order to investigate the sensitivity of the angle changes with respect to the manner of computation, the writer used the foregoing procedure for recomputing all cases of Example I. A maximum difference of 0.7% and an average difference for the eleven angle changes of only 0.37% was found. This comparison confirms the writer's opinion that the accuracy of all methods for computing angle changes is satisfactory for all practical purposes.

In conclusion, it would be valuable if Table 1 were extended to include the formulas for the angle changes corresponding to an external moment acting at any section of the beam, as such a loading condition is often important in frames carrying cranes.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

A SIMPLIFIED METHOD OF ANALYZING SUSPENSION BRIDGES

Discussion

BY PEDRO B. J. GRAVINA, AND G. S. RAMASWAMY

PEDRO B. J. GRAVINA.[13]—An interesting method of analysis, based on the deflection theory contrived to simplify the design of suspension bridges, is presented in this paper. Instead of the employing the cut-and-try method, it is possible, by means of the author's formula, to make a direct analysis, partly sacrificing exactness in favor of greater simplicity in design.

It is the opinion of the author (see Section 1) that, when employing the trigonometric series, "the computations are lengthy and involved." This statement is partly true for deflections, moments, and shears in the stiffening trusses; but it is not true in the case of horizontal tension in the cable, especially when the live load is uniformly distributed, as the convergence of the series is then rapid.

In order to evaluate the approximation of the formulas presented in the first part of the paper, it will be convenient to compare them with those derived directly from the deflection theory. In connection with some studies of the convergence of the series that appear through the method of the trigonometric series,[14] the writer has demonstrated the possibility of computing the value of H for a uniformly distributed live load by means of a simplified formula. In the special case of the symmetrical bridge with three variable spans, and with the load distributed between any two sections, distant k' and k'' from the left support (Fig. 4), the formula follows:

$$H = \left[p\left(\frac{8f}{l}\right)\frac{4\,l^2}{\pi^4}\frac{\bar{F}\left(\frac{k}{l}\right)}{1+\frac{\pi^2}{c^2\,l^2}} + p\left(\frac{8f_1}{l_1}\right)\frac{4\,l_1^2}{\pi^4}\frac{\bar{F}_{1l}\left(\frac{k}{l_1}\right)}{1+\frac{\pi^2}{c_1^2\,l_1^2}} + p\left(\frac{8f_1}{l_1}\right)\frac{4\,l_1^2}{\pi^4}\frac{\bar{F}_{1r}\left(\frac{k}{l_1}\right)}{1+\frac{\pi^2}{c_1^2\,l_1^2}}\right]\frac{1}{1+N'_c+\bar{N}'_t} \div l\,G + 2\,l_1\,G_1 \quad \ldots (106)$$

NOTE.—This paper by Ling-hi Tsien was published in September, 1948, *Proceedings*.

[13] Prof. of Bridge and Structural Eng., Polytechnic School, São Paulo Univ., São Paulo, Brazil.

[14] "Teoria das Pontes Pênseis," Pedro B. J. Gravina, editor, Polytechnic School of São Paulo Univ., São Paulo, Brazil, 1948, pp. 95–109.

in which, for the main span,

$$\bar{F}\left(\frac{k}{l}\right) = \cos\frac{k'\pi}{l} - \cos\frac{k''\pi}{l}; \qquad (k' < k'') \ldots\ldots\ldots (107a)$$

for the left side span,

$$\bar{F}_{1l}\left(\frac{k}{l_1}\right) = \cos\frac{k'\pi}{l_1} - \cos\frac{k''\pi}{l_1}; \qquad (k' < k'') \ldots\ldots\ldots (107b)$$

and, for the right side span,

$$\bar{F}_{1r}\left(\frac{k}{l_1}\right) = \cos\frac{k'\pi}{l_1} - \cos\frac{k''\pi}{l_1}; \qquad (k' < k'') \ldots\ldots\ldots (107c)$$

Also,

$$N'_c = \frac{H_w + H}{l\,G + 2\,l_1\,G_1}\,\frac{L_s}{E_c\,A_0} \ldots\ldots\ldots (108a)$$

$$\bar{N}'_t = \frac{H_w + H}{H\,(l\,G + 2\,l_1\,G_1)}\,\epsilon\,\Delta_t\,L_t \ldots\ldots\ldots (108b)$$

and

$$G = \frac{512}{\pi^4\left(1 + \frac{\pi^2}{c^2\,l^2}\right)}\left(\frac{f}{l}\right)^2 \ldots\ldots\ldots (109a)$$

and

$$G_1 = \frac{512}{\pi^4\left(1 + \frac{\pi^2}{c^2_1\,l^2_1}\right)}\left(\frac{f_1}{l_1}\right)^2 \ldots\ldots\ldots (109b)$$

In Eq. 106 the effect of temperature changes is represented only in the term $\bar{N}'_t$, as a correction to the approximate value of H. The functions $\bar{F}\left(\frac{k}{l}\right)$, $\bar{F}_{1l}\left(\frac{k}{l_1}\right)$, and $\bar{F}_{1r}\left(\frac{k}{l_1}\right)$, defined by Eqs. 107, are general and are valid for any loading condition, including, therefore, as special cases, the ones examined separately in Sections 3 to 8, inclusive, of the paper. As the length $\Delta l = k'' - k'$ decreases, the convergence of the series decreases, so that in the extreme case of a concentrated load it is no longer possible to consider only the first term of the series, as was done in the case of the distributed load; if this were done, the resulting error would be considerable. Consequently, Eq. 17*b* appears to be in much greater error than the other formulas in the paper that were used for the computation of the horizontal tension of the cable in the case of the distributed load.

Let

$$1 + \frac{\pi^2}{c^2\,l^2} = \bar{B} \ldots\ldots\ldots (110a)$$

and

$$1 + \frac{\pi^2}{c^2_1\,l^2_1} = \bar{B}_1 \ldots\ldots\ldots (110b)$$

and assume that only the main span is loaded, Eq. 106 can be written

$$H = \frac{p\left(\frac{8f}{l}\right)\frac{4l^2}{\pi^4}\frac{\bar{F}\left(\frac{k}{\bar{l}}\right)}{\bar{B}}}{\frac{512}{\pi^4}\left(\frac{f}{l}\right)^2\frac{l}{\bar{B}}}\frac{1}{1+N_c+\bar{N}_t+N_s} \quad \ldots\ldots(111)$$

in which

$$N_c = \frac{H_w + H}{l\,G}\frac{L_s}{E_c\,A_0} \quad \ldots\ldots(112a)$$

$$\bar{N}_t = \frac{H_w + H}{H\,l\,G}\epsilon\,\Delta_t\,L_t \quad \ldots\ldots(112b)$$

and

$$N_s = 2\frac{l_1\,G_1}{l\,G} \quad \ldots\ldots(112c)$$

If

$$C = N_c + \bar{N}_t + N_s \quad \ldots\ldots(113)$$

the value of H by Eqs. 111 and 112 will be

$$H = \frac{p\,l^2}{8f}\frac{\bar{F}\left(\frac{k}{\bar{l}}\right)}{2}\frac{1}{1+C} \quad \ldots\ldots(114)$$

Substituting the values of G and G_1 from Eqs. 109 in Eqs. 112,

$$N_c = \frac{0.189}{n^2}\frac{H_w}{E_c\,A_0}\frac{L_s}{l}\left(1+\frac{H}{H_w}\right)\bar{B} \quad \ldots\ldots(115a)$$

$$\bar{N}_t = \frac{0.189}{n^2}\frac{\epsilon\,\Delta_t\,L_t}{l}\left(1+\frac{H_w}{H}\right)\bar{B} \quad \ldots\ldots(115b)$$

and

$$N_s = 2\,r\,\frac{n^2_1}{n^2}\frac{\bar{B}}{\bar{B}_1} \quad \ldots\ldots(115c)$$

If the load occurs only on one of the side spans,

$$H = \frac{p\,l^2}{8f}\frac{n_1}{n}r^2\frac{\bar{B}}{\bar{B}_1}\frac{\bar{F}_1\left(\frac{k}{\bar{l}_1}\right)}{2}\frac{1}{1+C} \quad \ldots\ldots(116)$$

Let

$$B = 1 + \frac{\pi^2}{c^2_0\,l^2} \quad \ldots\ldots(117a)$$

and

$$B_1 = 1 + \frac{\pi^2}{c^2_{01}\,l^2_1} \quad \ldots\ldots(117b)$$

—it follows that

$$\bar{B} = \frac{1+\frac{H_w}{H}B}{1+\frac{H_w}{H}} \quad \ldots\ldots(118a)$$

and

$$\bar{B}_1 = \frac{1 + \frac{H_w}{H} B_1}{1 + \frac{H_w}{H}} \quad \ldots\ldots (118b)$$

Substituting Eqs. 118 in Eqs. 115 and 116,

$$N_c = \frac{0.189}{n^2} \frac{H}{E_c A_0} \frac{L_s}{n^2 l} \left(1 + \frac{H_w}{H} B \right) \quad \ldots\ldots (119a)$$

$$\bar{N}_t = \frac{0.189}{n^2} \frac{\epsilon \Delta_t L_t}{l} \left(1 + \frac{H_w}{H} B \right) \quad \ldots\ldots (119b)$$

$$N_s = 2 r \frac{n_1^2}{n^2} \frac{1 + \frac{H_w}{H} B}{1 + \frac{H_w}{H} B_1} \quad \ldots\ldots (119c)$$

and

$$H = \frac{p l^2}{8 f} \frac{n_1}{n} r^2 \frac{\bar{F}_1\left(\frac{k}{l_1}\right)}{2} \frac{1 + \frac{H_w}{H} B}{1 + \frac{H_w}{H} B_1} \frac{1}{1 + C} \quad \ldots\ldots (120)$$

By simple algebraic transformation, the following formulas can be derived from Eqs. 114 and 120:

$$H = \frac{p l^2}{8 f D} \left[\frac{\bar{F}\left(\frac{k}{l}\right)}{2} - T \right] \quad \ldots\ldots (121a)$$

and

$$H = \frac{p l^2}{8 f D} \left[\frac{S \bar{F}_1\left(\frac{k}{l_1}\right)}{2} - T \right] \quad \ldots\ldots (121b)$$

in which

$$S = r^2 \frac{n_1}{n} \frac{1 + \frac{H_w}{H} B}{1 + \frac{H_w}{H} B_1} \quad \ldots\ldots (122a)$$

$$T = \frac{0.189 \epsilon \Delta_t L_t B}{n^2 l q} \quad \ldots\ldots (122b)$$

$$D = 1 + N_c + N_s + N_t \quad \ldots\ldots (122c)$$

and

$$N_t = \frac{0.189}{n^2} \frac{\epsilon \Delta_t L_t}{l} \quad \ldots\ldots (122d)$$

Eqs. 119*a*, 119*c*, 121, and 122 are derived directly from the deflection theory and, therefore, afford an opportunity to appraise the equations for the variation of the horizontal component of the cable tension, and of the design constants, proposed by the author.

G. S. RAMASWAMY,[15] JUN. ASCE.—The already impressive array of literature on suspension bridges has been enhanced by the paper. The author's conclusion that the horizontal tension in the cable is practically independent of the ratio of the live load to the dead load and the degree of stiffness of the truss is provocative of further thought.

The writer wishes to confine his remarks to Part I of the paper, in which the author selects, as his point of departure, certain equations for the horizontal component of the cable tension which he has quoted from another source.[4] These equations were derived by energy methods; but they can be obtained more elegantly by the application of the Müller-Breslau principle.

The object is to determine the influence line for the horizontal component H of the cable tension for varying positions of a unit concentrated load traversing the span. The varying position of the unit concentrated load will be denoted by $n\,l$, in which n has values ranging from 0 to 1. The procedure is to give a unit deformation along the line of action of H. The resulting deflected shape of the stiffening truss will then give the influence line for H. The structure is thus made to draw its own influence line. If a unit horizontal tension is applied to the cable it will induce a uniform vertical load on the cable of intensity $\frac{8\,f}{l^2}$. The vertical uniform load, in turn, will exert a uniform pull on the stiffening truss, which will deflect as a hinged beam with a uniformly distributed load. The deflection curve of a hinged beam with a uniformly distributed load of intensity $\frac{8\,f}{l^2}$ is given by

$$\Delta = \frac{f\,l^2}{3\,E\,I}\,(n - 2\,n^3 + n^4) \ldots\ldots\ldots\ldots\ldots\ldots (123)$$

in which Δ is the deflection. The ordinates of the curve defined by Eq. 123 are not the influence line ordinates themselves, but are proportional to them, since a unit load was applied, instead of a unit deformation, along the line of action of H. The actual influence line curve can be obtained from Eq. 123 by the following artifice. Write the equation for the influence line as

$$H = A\,(n - 2\,n^3 + n^4) \ldots\ldots\ldots\ldots\ldots\ldots (124)$$

in which A is a constant to be determined by some known condition. When the entire span is covered by a uniformly distributed load of unit intensity, the horizontal tension can be obtained by integrating Eq. 124 and placing

[15] Instr., College of Eng., Kakinada, India.

[4] "Modern Framed Structures," by J. B. Johnson, C. W. Bryan, and F. E. Turneaure, John Wiley & Sons, Inc., New York, N. Y., 9th Ed., 1916, Pt. II, p. 219, Eq. 15.

$n = 1$. Performing the integration and substitution,

$$H_1 = \frac{A}{5} \quad \ldots\ldots (125a)$$

Since $H = \frac{l^2}{8f}$ for this particular condition of loading,

$$A = \frac{5\,l^2}{8\,f} \quad \ldots\ldots (125b)$$

Equating Eqs. 125*a* and 125*b*, the formula of the influence line for H is

$$H = \frac{5\,l^2}{8\,f}(n - 2\,n^3 + n^4) \quad \ldots\ldots (126)$$

Professor Tsien approximates the form of the influence line contained in Eq. 126 by a sine curve. Let the desired sine curve be

$$H = B \sin n\,\pi \quad \ldots\ldots (127a)$$

The area under this sine curve should be equal to $\frac{l^2}{8f}$. By applying this condition one obtains

$$H = \frac{l\,\pi}{16\,f} \sin n\,\pi \quad \ldots\ldots (127b)$$

Hence, the horizontal tension H created by a load P at a distance $n\,l$ from the left support is given by

$$H = \frac{P\,l\,\pi}{16\,f} \sin n\,\pi \quad \ldots\ldots (127c)$$

If $n\,l$ is replaced by k in Eq. 127*c*,

$$H = \frac{P\,l\,\pi}{16\,f} \sin \frac{k\,\pi}{l} \quad \ldots\ldots (127d)$$

Eq. 127*d* is the same as Eq. 17*b*, with the exception of a correction factor $\frac{1}{D}$.

The other cases considered by Professor Tsien can easily be deduced as corollaries by integrating either Eq. 127*b* or Eq. 127*d* and inserting appropriate limits.

The writer feels that the case of the concentrated load traversing the span is the fundamental one and should therefore be treated first; having once obtained the influence line for H, any distribution of live load can be solved conveniently.

Corrections for *Transactions:* In September, 1948, *Proceedings*, On page 1096, line 6, and in Eqs. 6 and 12 change "k" to "$\frac{k}{l}$" (no change in exponents); on page 1098, in Eq. 13, change the bracketed quantity to read "$\left[\frac{k^2}{2\,l^2}\left(\frac{2\,k^3}{l^3} - 5\,\frac{k^2}{l^2} + 5\right)\right]$"; on page 1105, line 5, change the last part of the

sentence to read "* * * magnitude of H, can therefore be neglected"; on page 1106, in Table 1, the equation for Case X, change "$\pm$" to "$\mp$"; on page 1109, line following Eq. 56*c*, change number 1 to letter l; on page 1113, section 31, lines 7 and 8, shorten last part of sentence to read "* * * of the error induced by this assumption"; on page 1123, section 47, line 4, change "Eq. 66" to "Eq. 64"; on page 1124, line 2 following Eq. 104, change "Eq. 103" to "Eq. 101*b*"; on page 1126, section 54, line 4, change "$\frac{45}{9.7}$" to "$\frac{54.6}{9.7}$"; on page 1127, section 56, line 14, change "section 54" to "section 53"; and, on page 1128, section 58, line 3, change "0.375 l" to "0.275 l"; and, line 13, change "Eq. 97" to "Eq. 96."

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

THE LOGICAL SYSTEM OF FREIGHT RATES

Discussion

BY OTTO MEYER

OTTO MEYER,[3] ASSOC. M. ASCE.—The proposed scheme for establishing a logical rate structure is well set up, and worthy of most serious consideration by legislative bodies. However, a major constitutional obstacle is the distinction between interstate and purely intrastate transportation.

Under one set of circumstances—namely, when the "fair" rate is more than the traffic can bear—an exception to the system should be made. An example might be a bulk agricultural product grown in a region so far from the market that it cannot be produced and sold at a profit without a favorable freight rate. Such a product might well be a keystone in the economy of the producing region. If it can be shown that the commodity in question will not be produced and shipped at all except with a freight rate concession, a special rate is justified if, and only if, the net earnings of the carrier giving the rate concession will be not less than they would be without giving it. The carrier would then not be subsidizing the commodity, nor would other commodities, as neither the shipping costs of other shippers nor the carrier's earnings would be adversely affected. On the contrary, the entire economy of the region would benefit. The writer believes, however, that special rates should never be allowed if increased shipping costs of other shippers or loss of earnings to the carrier would result.

NOTE.—This paper by Charles L. Hall was published in November, 1948, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: January, 1949, by Raphael H. Courland.

[3] Cons. Engr., Corvallis, Ore.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

INDUSTRIAL WASTES
A SYMPOSIUM

Discussion

BY S. LEARY JONES

S. LEARY JONES,[15] ASSOC. M. ASCE.—The necessity of establishing, sooner or later, permissible standards of pollution of surface waters was commented on by Professor Bloodgood. From an administrative viewpoint it is desirable to have such standards, but at present exact standards for most types of pollution are unknown. Considerable research and scientific investigation are certainly indicated and undoubtedly would be welcomed by all control agencies. In addition to helping to establish these standards, federal services can aid by assembling available data concerning the effects of various types of pollutants on streams.

The Tennessee Stream Pollution Control Board adopted the "General Criteria for the Definition and Control of Pollution in the Waters of Tennessee" in November, 1946, as a guide in determining the permissible conditions of waters with respect to pollution and also the preventive or corrective measures required to control pollution in various waters or in different sections of the same waters. The criteria recognize that the uses of waters and their capacities for sewage or waste disposal vary, depending on the volume of water, local needs, and other factors. The relative weight assigned to each type of use varies for different waters, and therefore the rigid application of uniform criteria of water quality is not considered desirable or reasonable.

Minimum criteria of water quality are established for the following: Solids, floating materials, and deposits; temperature; turbidity or color; taste or odor; acidity or alkalinity; hardness or mineral compounds; dissolved oxygen; bacteria; toxic substances; and other pollutants, including anything that may have been omitted in the other nine criteria. Interpretation is based upon the effect on the stream and the damage to reasonable and necessary uses. Numerical limits are included for only dissolved oxygen, bacteria, alkalinity, and

NOTE.—This Symposium was published in September, 1948, *Proceedings*.

[15] Prin. San. Engr., Stream Pollution Control, Tennessee Dept. of Public Health, Nashville, Tenn.

pH. Such criteria are flexible and well suited to the type of cooperative program approved by the board. The criteria are not suited to a cookbook type of control program which would eventually become unreasonable.

Stream pollution control in Tennessee is on a cooperative basis. No distinction is made between industrial wastes and sewage in so far as recommendations for correction are concerned. Each source of pollution is considered in relation to its effect on the receiving waters. No exceptions are provided in the law for either sewage or wastes, and the Tennessee Stream Pollution Control Board has not favored the classification of any stream as hopeless.

The Stream Pollution Control Act provides for a unified approach in that the membership of the board consists of two representatives from municipalities, two from industries, and the commissioners of the State Departments of Public Health, Conservation, and Agriculture. The commissioner of health is the chairman, and the Department of Public Health is the administrative agent for the board with the director, Division of Sanitary Engineering, serving as technical secretary.